Anal. Theory Appl.
Vol. 28, No. 3 (2012), 263–268
DOI : 10.3969/j.issn.1672-4070.2012.03.006

A CHARACTERIZATION FOR FRACTIONAL INTEGRALS ON GENERALIZED MORREY SPACES

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Received May 22, 2012

Abstract. This paper concerns with the fractional integrals, which are also known as the Riesz potentials. A characterization for the boundedness of the fractional integral operators on generalized Morrey spaces will be presented. Our results can be viewed as a refinement of Nakai's[7].

Key words: *fractional integrals, Morrey spaces*

AMS (2010) subject classification: 26A33, 42B35, 43A15, 47B38, 47G10

1 Introduction

For $0 < \alpha < d$, we define the fractional integral (also known as the Riesz potential) $I_\alpha f$ by

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} \mathrm{d}y, \qquad x \in \mathbf{R}^d,$$

for any suitable function f on $\mathbf{R}^d$. Clearly $I_\alpha f$ is well-defined for any locally bounded, compactly supported function f on $\mathbf{R}^d$. It is well-known that I_α is bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$, that is,

$$\|I_\alpha f : L^q\| \leq C \, \|f : L^p\|$$

if and only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d},$$

with $1 < p < \dfrac{d}{\alpha}$. This result was proved by Hardy and Littlewood[5,6] and Sobolev[10] around the 1930's. Further development on the subject can be found in [11, 12].

Next, let $\mathbf{R}^+ := (0,\infty)$. For $1 \le p < \infty$ and a suitable function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$, we define the generalized Morrey space $L^{p,\phi} = L^{p,\phi}(\mathbf{R}^d)$ to be the set of all functions $f \in L^p_{\rm loc}(\mathbf{R}^d)$ for which

$$\|f : L^{p,\phi}\| := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p \mathrm{d}y \right)^{1/p} < \infty.$$

Here the supremum are taken over all open balls $B = B(a,r)$ in $\mathbf{R}^d$ and $\phi(B) = \phi(r)$, where $r \in \mathbf{R}^+$. For certain functions ϕ, the spaces $L^{p,\phi}$ reduce to some classical spaces. For instance, if $\phi(r) = r^{(\lambda-d)/p}$, where $0 \le \lambda \le d$, then $L^{p,\phi}$ is the classical Morrey space $L^{p,\lambda}$. For a brief history of the Morrey space and related spaces, see [8]. For more recent results, see [9, 13] and the references therein.

In this short paper, we shall revisit Nakai's theorems on the fractional integrals on the generalized Morrey spaces[7]. In particular, we find that the sufficient condition imposed by Nakai for the boundedness of the operator turns out to be necessary. In other words, we obtain a characterization for which the fractional integral operators are bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

2 Main Results

Let us begin with some assumptions and relevant facts that follow. As customary, the letters C, C_i, C_p and $C_{p,q}$ denote positive constants, which may depend on the parameters such as α, p,q and the dimension d of the ambient space, but not on the function f or the variable x. These constants may vary from line to line.

In the definition of $L^{p,\phi}$, the function ϕ is assumed to satisfy the following conditions:

$$\begin{aligned} \phi \text{ is almost decreasing} \quad &: \quad t \le r \Rightarrow \phi(r) \le C_1 \phi(t); \\ r^d \phi(r)^p \text{ is almost increasing} \quad &: \quad t \le r \Rightarrow t^d \phi(t)^p \le C_2 r^d \phi(r)^p, \end{aligned}$$

with $C_1,\ C_2 > 0$ being independent of r and t. These two conditions imply that

$$\phi \text{ satisfies the doubling condition} \quad : \quad 1 \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{C_3} \le \frac{\phi(t)}{\phi(r)} \le C_3,$$

for some $C_3 > 0$ (which is also independent of r and t). Throughout this paper, we shall always assume that ϕ satisfies these conditions.