NON-ORTHOGONAL $P$-WAVELET PACKETS ON
THE HALF-LINE

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Received May 23, 2011; Revised Dec. 26, 2012

Abstract. In this paper, the notion of $p$-wavelet packets on the positive half-line $\mathbb{R}^+$ is introduced. A new method for constructing non-orthogonal wavelet packets related to Walsh functions is developed by splitting the wavelet subspaces directly instead of using the low-pass and high-pass filters associated with the multiresolution analysis as used in the classical theory of wavelet packets. Further, the method overcomes the difficulty of constructing non-orthogonal wavelet packets of the dilation factor $p > 2$.

Key words: $p$-Multiresolution analysis, $p$-wavelet packets, Riesz basis, Walsh function, Walsh-Fourier transform

AMS (2010) subject classification: 42C40, 42C10, 42C15

1 Introduction

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat\cite[16]{Mallat}. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbb{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups\cite[5]{Dahlke}, Lang\cite[14]{Lang} constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Daubechies\cite[6]{Daubechies} via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. On the otherhand, Jiang et al.\cite[13]{Jiang} pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Subsequently, the tight wavelet frames on local
fields were constructed by Li and Jiang in \cite{15}. Farkov \cite{7} extended the results of Lang \cite{14} on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group $G_p$ which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Concerning the construction of wavelets on a half-line, Farkov \cite{8} has given the general construction of all compactly supported orthogonal $p$-wavelets in $L^2(\mathbb{R}^+)$ and proved necessary and sufficient conditions for scaling filters with $p^n$ many terms ($p, n \geq 2$) to generate a $p$-MRA analysis in $L^2(\mathbb{R}^+)$. These studies were continued by Farkov and his colleagues in \cite{9,10}, where they have given some new algorithms for constructing the corresponding biorthogonal and nonstationary wavelets related to the Walsh functions on the positive half-line $\mathbb{R}^+$. On the other hand, Shah and Debnath \cite{21} have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system \{2^{j/2}�(2^jx \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\} to be a frame for $L^2(\mathbb{R}^+)$. It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet $ψ$ is band limited, then the measure of the supp of $(ψ_j,k) \hat{\psi}$ is $2^j$-times that of supp $\hat{ψ}$. To overcome this disadvantage, Coifman et al. \cite{4} constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application.

Let $ϕ(x)$ and $ψ(x)$ be the scaling function and the wavelet function associated with a multiresolution analysis $\{V_j\}_{j \in \mathbb{Z}}$. Let $W_j$ be the corresponding wavelet subspaces:

$$W_j = \text{span}\{ψ_{j,k} : k \in \mathbb{Z}\}.$$  

Using the low-pass and high-pass filters associated with the MRA, the space $W_j$ can be split into two orthogonal subspaces, each of them can further be split into two parts. Repeating this process $j$ times, $W_j$ is decomposed into $2^j$ subspaces each generated by integer translates of a single function. If we apply this to each $W_j$, then the resulting basis of $L^2(\mathbb{R})$ which will consist of integer translates of a countable number of functions, will give a better frequency localization. This basis is called the wavelet packet basis. To describe this more formally, we introduce a parameter $n$ to denote the frequency. Set $ω_0 = ϕ$ and define recursively

$$ω_{2n}(x) = \sum_{k \in \mathbb{Z}} h_k ω_n(2x - k), \quad ω_{2n+1}(x) = \sum_{k \in \mathbb{Z}} g_k ω_n(2x - k),$$

where $\{h_k\}_{k \in \mathbb{Z}}$ and $\{g_k\}_{k \in \mathbb{Z}}$ are the low-pass filter and high-pass filter corresponding to $ϕ(x)$ and $ψ(x)$, respectively. Chui and Li \cite{2} generalized the concept of orthogonal wavelet packets to