Analysis in Theory and Applications
Anal. Theory Appl., Vol. **29**, No. 1 (2013), pp. 1-11

DOI: 10.4208/ata.2013.v29.n1.1

More on Fixed Point Theorem of $\{a,b,c\}$-Generalized Nonexpansive Mappings in Normed Spaces

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Received 15 May 2008

Abstract. Let X be a weakly Cauchy normed space in which the parallelogram law holds, C be a bounded closed convex subset of X with one contracting point and T be an $\{a,b,c\}$-generalized-nonexpansive mapping from C into C. We prove that the infimum of the set $\{\|x-T(x)\|\}$ on C is zero, study some facts concerning the $\{a,b,c\}$-generalized-nonexpansive mapping and prove that the asymptotic center of any bounded sequence with respect to C is singleton. Depending on the fact that the $\{a,b,0\}$-generalized-nonexpansive mapping from C into C has fixed points, accordingly, another version of the Browder's strong convergence theorem for mappings is given.

Key Words: Fixed point theorem, $\{a,b,c\}$-generalized-nonexpansive mapping, asymptotic center, Browder's strong convergence Theorem.

AMS Subject Classifications: 42B25, 42B20

1 Introduction

The following Banach contraction principle is a basic theorem of fixed point theory.

Theorem 1.1. *If X is a complete metric space and T is a r-contraction mapping from X into itself, then T has a unique fixed point $y\in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x\in X$.*

Mathematicians in the field of fixed point theory try to improve the result of this theorem in which changing the r-contractivity assumption imposed on the given mapping or reduce the completeness condition on the given topological space.

F. Edelstein proved that if X is a compact metric space and T is a contractive mapping from X into itself, then T has a unique fixed point $y\in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x\in X$, see, e.g., [5]. More explicitly,

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changing the r-contractivity condition imposed on the mapping to the contractivity condition on the mapping needs to strengthen the completeness condition to compactness condition on the metric space.

One can see that nonexpansive mappings require compact Hilbert sapce [4]. Also proper convex semi continuous mappings are used in Caristi's Fixed Point Theorem [5].

The nonexpansive mapping may not have fixed point even on uniformly convex (not bounded) Banach spaces in general. In [6], it is proved that the existence of a unique fixed point of the contraction mapping defined on a closed convex subset of a weakly Cauchy normed space. In [2], it is proved the existence of fixed point of a nonexpansive mapping defined on a bounded closed convex subset C of a weakly Cauchy normed space X in which parallelogram law holds. In [7], it gave a generalization of Banach contraction principle in two directions, and proved the existence of a unique fixed point of $\{a,b,c\}$-generalized-contraction mappings defined on the closed subset of a weakly Cauchy normed space.

The problem whether the parallelogram law holds in $\{a,b,c\}$-generalized-nonexpansive mappings defined on a bounded closed convex subset C of a weakly Cauchy normed space X in which the parallelogram law holds or the problem even in uniformly convex Banach space is open and has no affirmative solution.

In this paper, we will extend my study in this field, introduce the concept of $\{a,b,c\}$-generalized-nonexpansive mapping defined on a bounded closed convex subset of a weakly Cauchy normed space and show that $\inf\{\|x-T(x)\|: x\in C\}=0$ provided the contracting point is in C.

We will also show that the asymptotic center of any bounded sequence with respect to a closed convex subset of a weakly normed space in which the parallelogram law holds is singleton. Depending on this fact the $\{a,b,0\}$-generalized-nonexpansive mapping from C into C has fixed points, accordingly, another version of the Browder's strong convergence theorem for mappings is given.

2 Notations and basic definitions

Let X be a linear space and $f: X\to(-\infty,\infty]$ be a function from X into $(-\infty,\infty]$. Then [8]:

(1) f is said to be proper if and only if there is $x\in X$ such that $f(x)<\infty$;

(2) f is said to be lower semicontinuous if and only if the set $\{x\in X: f(x)\le\alpha\}$ is a closed convex subset of X for any real number α;

(3) f is said to be convex if and only if $f(tx+(1-t)y\le tf(x)+91-t)f(y)$ for any $x,y\in X$ and $t\in[0,1]$.

Let X be a normed space and A be a mapping from X into itself. Then

(i) A is said to be an r-contraction, a real number r with, $0\le r<1$, if and only if

$$\|A(x)-A(y)\|\le r\|x-y\| \text{ for every } x \text{ and } y, \quad x,y\in X.$$