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Weak Type Estimates for Intrinsic Square Functions on Weighted Morrey Spaces

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Abstract. In this paper, we will obtain the weak type estimates of intrinsic square functions including the Lusin area integral, Littlewood-Paley g-function and g^*_λ-function on the weighted Morrey spaces $L^{1,\kappa}(w)$ for $0<\kappa<1$ and $w\in A_1$.

Key Words: Intrinsic square function, weighted Morrey space, A_p weight.

AMS Subject Classifications: 42B25, 42B35

1 Introduction and main results

Let $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times(0,\infty)$ and $\varphi_t(x)=t^{-n}\varphi(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x,t)=P_t*f(x)$ is the Poisson integral of f, where

$$P_t(x)=c_n\frac{t}{(t^2+|x|^2)^{(n+1)/2}}$$

denotes the Poisson kernel in $\mathbb{R}^{n+1}_+$. Then we define the classical square function (Lusin area integral) $S(f)$ by (see [16,17])

$$S(f)(x)=\left(\iint_{\Gamma(x)}\big|\nabla u(y,t)\big|^2t^{1-n}dydt\right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x)=\Big\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\Big\}$$

and

$$\big|\nabla u(y,t)\big|^2=\left|\frac{\partial u}{\partial t}\right|^2+\sum_{j=1}^n\left|\frac{\partial u}{\partial y_j}\right|^2.$$

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Similarly, we can define the cone of aperture β for any $\beta>0$:

$$\Gamma_\beta(x)=\Big\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<\beta t\Big\}$$

and the corresponding square function

$$S_\beta(f)(x)=\bigg(\iint_{\Gamma_\beta(x)}\big|\nabla u(y,t)\big|^2 t^{1-n}dydt\bigg)^{1/2}.$$

The Littlewood-Paley g-function (could be viewed as a "zero-aperture" version of $S(f)$) and the g^*_λ-function (could be viewed as an "infinite aperture" version of $S(f)$) are defined respectively by

$$g(f)(x)=\bigg(\int_0^\infty\big|\nabla u(x,t)\big|^2 t dt\bigg)^{1/2}$$

and

$$g^*_\lambda(f)(x)=\bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\big|\nabla u(y,t)\big|^2 t^{1-n}dydt\bigg)^{1/2},\quad \lambda>1.$$

The modern (real-variable) variant of $S_\beta(f)$ can be defined in the following way (here we drop the subscript β if $\beta=1$). Let $\psi\in C^\infty(\mathbb{R}^n)$ be real, radial, have support contained in $\{x:|x|\le 1\}$, and $\int_{\mathbb{R}^n}\psi(x)dx=0$. The continuous square function $S_{\psi,\beta}(f)$ is defined by (see, e.g., [2,3])

$$S_{\psi,\beta}(f)(x)=\bigg(\iint_{\Gamma_\beta(x)}\big|f*\psi_t(y)\big|^2\frac{dydt}{t^{n+1}}\bigg)^{1/2}.$$

In 2007, Wilson [25] introduced a new square function called intrinsic square function which is universal in a sense (see also [26]). This function is independent of any particular kernel ψ, and it dominates pointwise all the above-defined square functions. On the other hand, it is not essentially larger than any particular $S_{\psi,\beta}(f)$. For $0<\alpha\le 1$, let $\mathcal{C}_\alpha$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x\in\mathbb{R}^n:|x|\le 1\}$, $\int_{\mathbb{R}^n}\varphi(x)dx=0$, and for all $x,x'\in\mathbb{R}^n$,

$$|\varphi(x)-\varphi(x')|\le|x-x'|^\alpha.$$

For $(y,t)\in\mathbb{R}^{n+1}_+$ and $f\in L^1_{loc}(\mathbb{R}^n)$, we set

$$A_\alpha(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\alpha}\big|f*\varphi_t(y)\big|=\sup_{\varphi\in\mathcal{C}_\alpha}\bigg|\int_{\mathbb{R}^n}\varphi_t(y-z)f(z)dz\bigg|.$$

Then we define the intrinsic square function of f of order α by the formula

$$\mathcal{S}_\alpha(f)(x)=\bigg(\iint_{\Gamma(x)}\Big(A_\alpha(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\bigg)^{1/2}.$$