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The Boundedness of Littlewood-Paley Operators with Rough Kernels on Weighted $(L^q,L^p)^\alpha(\mathbf{R}^n)$ Spaces

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Abstract. In this paper, we shall deal with the boundedness of the Littlewood-Paley operators with rough kernel. We prove the boundedness of the Lusin-area integral $\mu_{\Omega,s}$ and Littlewood-Paley functions μ_Ω and μ_λ^* on the weighted amalgam spaces $(L^q_\omega,L^p)^\alpha(\mathbf{R}^n)$ as $1<q\le\alpha<p\le\infty$.

Key Words: Littlewood-Paley operator, weighted amalgam space, rough kernel, A_p weight.

AMS Subject Classifications: 42B25, 42B20

1 Introduction and main result

Let $1\le p,\ q\le\infty$, a function $f\in L^q_{loc}(\mathbf{R}^n)$ is said to be in the amalgam spaces $(L^q,L^p)(\mathbf{R}^n)$ of $L^q(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n)$ if $\|f(\cdot)\chi_{B(y,1)(\cdot)}\|_q$ belongs to $L^p(\mathbf{R}^n)$, where $B(y,r)$ denotes the open ball with center y and radius r and $\chi_{B(y,r)}$ is the characteristic function of the ball $B(y,r)$, $\|\cdot\|_q$ is the usual Lebesgue norm in $L^q(\mathbf{R}^n)$.

$$\|f\|_{q,p}=\left(\int_{\mathbf{R}^n}\|f\chi_{B(y,1)}\|_q^p\,\mathrm{d}y\right)^{1/p}$$

is a norm on $(L^q,L^p)(\mathbf{R}^n)$ under which it is a Banach space with the usual modification when $p=\infty$.

Amalgam spaces were first introduced by N. Winer in 1926. But its systematic study goes back to the work of Holland [12]. We refer the reader to see the survey paper of Fournier and Stewart [10] for more information about these spaces. We recall some of their properties. Let $1\le q,\ p\le\infty$, the following relations hold:

1. $(L^q,L^q)(\mathbf{R}^n)=L^q(\mathbf{R}^n)$;

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2. $L^q(\mathbf{R}^n)\cup L^p(\mathbf{R}^n)\subset(L^q,L^p)(\mathbf{R}^n)$ if $q\le p$;

3. $(L^q,L^p)(\mathbf{R}^n)\subset L^q(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$ if $q\ge p$.

Since the properties above, the amalgam spaces $(L^q,L^p)(\mathbf{R}^n)$ are interested especially when $q\le p$. Let δ_r^α denotes the dilation operator defined by $\delta_r^\alpha: f\longmapsto r^{\frac{n}{\alpha}}f(r\cdot)$ for any $r>0$ and $\alpha>0$. It is easy to see that

$$f\in(L^q,L^p)(\mathbf{R}^n)\Leftrightarrow\|\delta_r^\alpha f\|_{q,p}<\infty,\quad\text{for any } r>0,\ \alpha>0,$$

where

$$\begin{aligned}\|\delta_r^\alpha f\|_{q,p}&=r^{n(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\left(\int_{\mathbf{R}^n}\|f\chi_{B(y,r)}\|_q^p\mathrm{d}y\right)^{1/p}\\&\approx\left[\int_{\mathbf{R}^n}(|B(y,r)|^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\|f\chi_{B(y,r)}\|_q)^p\mathrm{d}y\right]^{1/p},\end{aligned}$$

and $|B(y,r)|$ denotes the Lebesgue measure of $B(y,r)$.

For $1\le q,p,\ \alpha\le\infty$, let

$$(L^q,L^p)^\alpha(\mathbf{R}^n)=\left\{f:\|f\|_{q,p,\alpha}=\sup_{r>0}\|\delta_r^\alpha f\|_{q,p}<\infty\right\}.$$

The above spaces are generalized in the content of spaces of homogeneous type in the sense of Coifman and Weiss (see [8]). The spaces $(L^q,L^p)^\alpha(\mathbf{R}^n)$ were first introduced by Fofana in [9] and it was proved that these spaces are non trivial if and only if $q\le\alpha\le p$. It was proved in [1,9], for $1\le q<\alpha$ fixed and p going from α to ∞, then they form a chain of distinct Babach spaces beginning with the Lebesgue space $L^\alpha(\mathbf{R}^n)$ and ending by the classical Morrey's space $L^{q,\kappa}(\mathbf{R}^n)=(L^q,L^\infty)^\alpha(\mathbf{R}^n)$, $\kappa=\frac{1}{q}-\frac{1}{\alpha}$. These spaces and their properties have been extended in the content of homogeneous groups in [6] (see also [7]). We recall that many classical results established in the content of $(L^q,L^p)^\alpha(\mathbf{R}^n)$ spaces. For example, Hölder and Young inequalities are just a consequence of their analog in Lebesgue spaces [9]. The Hardy-Littlewood-Sobolve inequality for fractional integrals has been generalized to this case in [1,5]. The boundedness of intrinsic square function and the corresponding commutators generated by bounded mean oscillation functions on a family of weighted subspaces of Morrey spaces were established in [7].

Let $S^{n-1}=\{x\in\mathbf{R}^n:|x|=1\}$ denotes the unit sphere on $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma=\mathrm{d}\sigma(x')$. Let $\Omega\in L^{q_1}(S^{n-1})$ with $1<q_1\le\infty$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0,\tag{1.1}$$

where $x'=x/|x|$ for any $x\neq0$. The Lusin-area integral $\mu_{\Omega,s}$ is defined by

$$\mu_{\Omega,s}(f)(x)=\left(\int\int_{\Gamma(x)}\Big|\frac{1}{t}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{1/2},\tag{1.2}$$