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Volterra Integral Equation of Hermite Matrix Polynomials

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Abstract. The primary purpose of this paper is to present the Volterra integral equation of the two-variable Hermite matrix polynomials. Moreover, a new representation of these matrix polynomials are established here.

Key Words: Hermite matrix polynomials, three terms recurrence relation and Volterra integral equation.

AMS Subject Classifications: 15A15, 33C45, 42C05, 45D05

1 Introduction

In [9], the Laguerre and Hermite matrix polynomials are introduced as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. The Hermite matrix polynomials $H_n(x,A)$, have been introduced and studied in [7,8], where $H_n(x,A)$ involves a parameter $A \in \mathbb{C}^{N\times N}$. Indeed, all eigenvalues of the matrix A lie on the open right-hand half of the complex plane.

Some properties of series expansions and the bounds of the $H_n(x,A)$ are given in [3–5]. Moreover, two generalizations of $H_n(x,A)$ are given in [2,12]. Recently, an efficient method for computing matrix exponentials based on Hermite matrix polynomial expansions has been presented in [11].

The aim of this paper is to provide some results on the two-variable Hermite matrix polynomials and introduce the Volterra integral equation of these matrix polynomials. The structure of this paper is the following. Section 2 summarizes previous results of the two-variable Hermite matrix polynomials and includes a new property of these matrix polynomials. In Section 3, we construct the Volterra integral equation of the two-variable Hermite matrix polynomials.

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Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, we denote by $\sigma(A)$ the spectrum of A or the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the property of the matrix functional calculus [6, p. 558], it follows that

$$f(A)g(A)=g(A)f(A). \tag{1.1}$$

In what follows, the matrices I and θ in $\mathbb{C}^{N\times N}$ denote the matrix identity and the zero matrix of order N, respectively.

2 The two-variable Hermite matrix polynomials

In this section, for the sake of clarity in the presentation of the following results we recall some properties of the two-variable Hermite matrix polynomials which have been established in [2]. Moreover, some new properties concerning these matrix polynomials are given here.

If D_0 is the complex plane cut along the negative real axis and Log(z) denotes the principle logarithm of z, then $z^{1/2}$ represents $\exp(\frac{1}{2}Log(z))$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A. Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$Re(\lambda)>0 \quad \text{for all } \lambda\in\sigma(A). \tag{2.1}$$

Then the two-variable Hermite matrix polynomial **[2VHMPs]** are defined in [2, p. 84] by

$$H_n(x,y,A)=n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k y^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}, \tag{2.2}$$

where $\lfloor a\rfloor$ is the standard floor function which maps a real number a to its next smallest integer. According to [2, Theorem 7], it follows that

$$(x\sqrt{2A})^n=\exp\Big(y(2A)^{-1}\frac{\partial^2}{\partial x^2}\Big)H_n(x,z,A). \tag{2.3}$$

Note that the **2VHMPs** are the solutions of the following matrix differential equation:

$$\Big[y\frac{\partial^2}{\partial x^2}-xA\frac{\partial}{\partial x}+nA\Big]H_n(x,y,A)=\theta; \quad n\geq 0, \tag{2.4}$$

and satisfy the three terms recurrence relationship:

$$H_n(x,y,A)=x\sqrt{2A}H_{n-1}(x,y,A)-2(n-1)yH_{n-2}(x,y,A); \quad n\geq 2 \tag{2.5}$$