Endpoint Estimates for Hardy Operator’s Conjugate Operator with Power Weight on $n$-Dimensional Space

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Abstract. In this paper, we establish two integral inequalities for Hardy operator’s conjugate operator at the endpoint on $n$-dimensional space. The operator $H^*_n$ is bounded from $L^1_x(G^n)$ to $L^q_{x^q}(G^n)$ with the bound explicitly worked out and the similar result holds for $H^*_n$.

Key Words: Conjugate operator, power weight, endpoint estimate.

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1 Introduction

Let $f$ be a non-negative integrable function on $G := (0, \infty)$. The classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t)dt,$$

and its conjugate operator

$$H^*f(x) := \int_x^\infty \frac{f(t)}{t}dt$$

for all $x > 0$.

For $n$-dimensional case with $n \geq 2$, Hardy operators can be defined on product space as

$$H_nf(x) := \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \cdots, t_n)dt_1 \cdots dt_n, \quad (1.1)$$

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and its conjugate operator defined as

$$H_n^* f(x) := \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(t_1, \cdots, t_n)}{t_1 \cdots t_n} dt_1 \cdots dt_n,$$  \hspace{1cm} (1.2)

for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{G}^n = (0, \infty)^n$, where $f$ is any measurable function on $\mathbb{G}^n$.

Another definition is given by Christ and Grafakos in [2] as follows

$$\mathcal{H}_n f(x) = \frac{1}{\omega_n |x|^n} \int_{|y|<|x|} f(y) \, dy,$$  \hspace{1cm} (1.3)

and

$$\mathcal{H}_n^* f(x) = \frac{1}{\omega_n |y|^n} \int_{|y|>|x|} \frac{f(y)}{|y|^n} \, dy,$$  \hspace{1cm} (1.4)

for $x \in \mathbb{R}^n \setminus \{0\}$, where $f$ is any measurable function on $\mathbb{R}^n$ and $\omega_n = \pi^{n/2} / \Gamma(1 + n/2)$ is the volume of the unit ball in $\mathbb{R}^n$.

For the case $1 < p \leq q < \infty$, the boundedness of the operators $H_n$ and $H_n^*$ from $L^p_p(\mathbb{G}^n)$ to $L^q_p(\mathbb{G}^n)$ were discussed in many papers (cf. [1, 13, 15, 17]). The estimates on the endpoint for the operator $H_n$ and $\mathcal{H}_n$ were systematically studied in [18]. It should be pointed out that the operator $H_n$, $n \geq 2$, defined by (1.1) fails to be of weak type of $(1, 1)$, however, $H_n$ is bounded from $L^1_p(\mathbb{G}^n)$ to $L^q_p(\mathbb{G}^n)$ for arbitrary $n \in \mathbb{N}$. This shows that some power weight can change the boundedness of the operator $H_n$ on the endpoint. Motivated by the idea of the reference [18], it is natural for us to discuss the boundedness of the operator $H_n^*$ on the endpoint.

The purpose of this paper is to establish the boundedness of the operators $H_n^*$ and $\mathcal{H}_n^*$ on the endpoint.

Throughout the paper, we have the following notations. For two $n$-dimensional vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, $\alpha \cdot \beta = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_n \beta_n$, $\alpha < \beta$ means each $\alpha_i < \beta_i$, $i = 1, \cdots, n$, and $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, $x \in \mathbb{G}^n$. For some bold type figures and letters, we have $1 = (1, \cdots, 1)$, $p = (p, \cdots, p)$. It is clear that $x^1 = x_1 \cdots x_n$.

Now we first formulate our main results as follows.

**Theorem 1.1.** Suppose that $f$ is any non-negative measurable function on $\mathbb{G}^n$ and $1 \leq q < \infty$. If $\alpha$ and $\beta$ are two $n$-tuples in $\mathbb{R}^n$ such that $\alpha + 1 > 0$ and $\beta + 1 = q(\alpha + 1)$, then the following inequality

$$\left( \int_{\mathbb{G}^n} (H_n^* f(x))^q x^\beta \, dx \right)^\frac{1}{q} \leq \left( \prod_{i=1}^{n} \frac{1}{(\beta_i + 1)} \right)^\frac{1}{q} \int_{\mathbb{G}^n} f(x) x^\alpha \, dx$$  \hspace{1cm} (1.5)

holds for the operator $H_n^*$ defined by (1.2), that is, $H_n^*$ is bounded from $L^1_{x^n}(\mathbb{G}^n)$ to $L^q_{x^\beta}(\mathbb{G}^n)$ with the norm of $H_n^*$ satisfying

$$\|H_n^*\| \leq \left( \prod_{i=1}^{n} \frac{1}{(\beta_i + 1)} \right)^\frac{1}{q}.$$