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A Note on a Theorem of J. Szabados

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Abstract. In this note, we establish a companion result to the theorem of J. Szabados on the maximum of fundamental functions of Lagrange interpolation based on Chebyshev nodes.

Key Words: Lagrange interpolation, Chebyshev polynomial, fundamental function of interpolation.

AMS Subject Classifications: 41A05, 41A10

1 Introduction

Let $T_n(x)=\cos(n\arccos x)$ be the Chebyshev polynomial of degree n with the roots

$$x_{k,n}=\cos t_{k,n},\quad t_{k,n}=\frac{2k-1}{2n}\pi,\quad k=1,2,\cdots,n,$$

and let

$$l_{k,n}(x)=\frac{(-1)^{k-1}\cos nt\sin t_{k,n}}{n(\cos t-\cos t_{k,n})},\quad x=\cos t,\quad k=1,2,\cdots,n, \tag{1.1}$$

be the fundamental polynomials of Lagrange interpolation based on the Chebyshev nodes. Setting

$$\begin{aligned}
&l_{k,n}=\max_{|x|\le 1} l_{k,n}(x), && k=1,2,\cdots,n,\\
&M_n(x)=\max_{1\le k\le n} l_{k,n}(x), && |x|\le 1,\\
&\overline{M}_n=\max_{|x|\le 1} M_n(x), && \underline{M}_n=\min_{|x|\le 1} M_n(x),\\
&\overline{M}_n^*=\max_{1\le k\le n} l_{k,n}, && \underline{M}_n^*=\min_{1\le k\le n} l_{k,n},
\end{aligned}$$

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it is easy to see that

$$\underline{M}_n \le \overline{M}_n \le \overline{M}_n^*,$$

and

$$\underline{M}_n^* \le \overline{M}_n.$$

In [1], Erdös and Grünwald proved the following theorem.

Theorem 1.1. *We have*

$$|l_{k,n}(x)| < \frac{4}{\pi}, \quad |x| \le 1, \quad 1 \le k \le n, \quad n = 1,2,\cdots. \tag{1.2}$$

Moreover,

$$\lim_{n\to\infty} l_{1,n}(1) = \lim_{n\to\infty} l_{n,n}(-1) = \frac{4}{\pi}. \tag{1.3}$$

It follows from Theorem 1.1 that

$$\lim_{n\to\infty} \overline{M}_n = \lim_{n\to\infty} \overline{M}_n^* = \frac{4}{\pi}.$$

In [2], J. Szabados proved the following theorem.

Theorem 1.2. *We have*

$$\lim_{n\to\infty} \underline{M}_n = \frac{2}{\pi} \cos \frac{2-\sqrt{3}}{2} \pi = 0.580\cdots. \tag{1.4}$$

It is natural to ask that which of $\underline{M}_n^*$ and $\underline{M}_n$ is bigger and what is the behavior of $\underline{M}_n^*$? In this note we prove the following theorem.

Theorem 1.3. *We have*

$$\lim_{n\to\infty} \underline{M}_n^* = 1. \tag{1.5}$$

2 Proof of Theorem 1.3

For convenience, we denote $t_{k,n}$, $x_{k,n}$, $l_{k,n}(x)$ and $l_{k,n}$ by t_k, x_k, $l_k(x)$, and l_k respectively and denote $l_k(\cos t)$ by $f_k(t)$, $k = 1,2,\cdots,n$. In order to prove Theorem 1.3, we need the following lemmas.

Lemma 2.1. *For* $k = 2,3,\cdots,[(n+1)/2]$, $n > 2$, $t \in [0, t_{k-1}] \cap [t_{k+1}, \pi]$, *we have*

$$|f_k(t)| \le \frac{2}{\pi}. \tag{2.1}$$