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Fixed Point Theorem of $\{a,b,c\}$ Contraction and Nonexpansive Type Mappings in Weakly Cauchy Normed Spaces

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Abstract. Let C be a closed convex weakly Cauchy subset of a normed space X. Then we define a new $\{a,b,c\}$ type nonexpansive and $\{a,b,c\}$ type contraction mapping T from C into C. These types of mappings will be denoted respectively by $\{a,b,c\}$-$n$type and $\{a,b,c\}$-$c$type. We proved the following:

1. If T is $\{a,b,c\}$-$n$type mapping, then $\inf\{\|T(x)-x\|:x\in C\}=0$, accordingly T has a unique fixed point. Moreover, any sequence $\{x_n\}_{n\in\mathcal{N}}$ in C with $\lim_{n\to\infty}\|T(x_n)-x_n\|=0$ has a subsequence strongly convergent to the unique fixed point of T.
2. If T is $\{a,b,c\}$-$c$type mapping, then T has a unique fixed point. Moreover, for any $x\in C$ the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ has subsequence strongly convergent to the unique fixed point of T.

This paper extends and generalizes some of the results given in [2,4,7] and [13].

Key Words: Fixed point, generalized type of contaction and nonexpansive mappings, normed space.

AMS Subject Classifications: 42B25, 42B20

1 Introduction

Let C be a closed convex subset of a normed space X and T be a mapping from C into C which satisfies $\|T(x)-T(y)\|\le a\|x-y\|+b\|y-T(y)\|+c\|x-T(x)\|$ for all $x,y\in C$ and for some real numbers $a,b,c\in[0,1]$.

When $0<a<1$, $b=c=0$, T becomes a contraction mapping. If X is complete, S. Banach gave his famous Banach contraction mappings principle, namely, T has a unique fixed point.

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When $a=1$, $b=c=0$, T becomes a nonexpansive mapping, if C is a bounded closed convex subsets of a Banach space X, W. A. Kirk proved fixed point theorems concerning this type mappings, [7].

Recently, the existence of fixed points of T when the domain of T is unbounded discussed in [5].

When $a=0$, we have the Kannan maps, see [6].

When $a+b+c<1$ a unique fixed point of mappings T defined on a closed convex subset of a weakly Cauchy normed space is proved [2].

Theorem 1.1 (see [2]). *Let X be a normed space, C be a closed convex and weakly Cauchy subset of X and T be a mapping from C into C which satisfies $\|T(x)-T(y)\|\le a\|x-y\|+b\|y-T(y)\|+c\|x-T(x)\|$ for all $x,y\in C$ and for some real numbers $a,b,c\in[0,1]$ with $a+b+c<1$. Then T has a unique fixed point $y\in C$.*

When $0<a<1$, $b,c\ge 0$, and $a+b+c=1$, T becomes Gregus type mapping, M. Gregus proved the existence of a unique fixed point of such mappings provided that C is closed convex subset of a Banach space X [4].

Theorem 1.2 (see [4]). *Let C be a closed convex subset of a Banach space X and T be a mapping from C into C which satisfies $\|T(x)-T(y)\|\le a\|x-y\|+b\|y-T(y)\|+c\|x-T(x)\|$ for all $x,y\in C$ and for some real numbers $a,b,c\in[0,1]$ with $0<a<1$ and $a+b+c=1$. Then T has a unique fixed point $y\in C$.*

More general contraction type mapping was given in [3,8], and [9]. It is proved that

Theorem 1.3 (see [3]). *Let (X,d) be a complete metric space and T be a mapping from X into X which satisfies $d(T(x),T(y))\le ad(x,y)+bd(y,T(y))+cd(x,T(x))+ed(T(x),y)+fd(T(y),x)$ for all $x,y\in X$ and for some real numbers a, b, c, e, and $f\in[0,1]$ with $a+b+c+e+f<1$. Then T has a unique fixed point.*

When $a+b+c=1$, T becomes $\{a,b,c\}$-Generalized nonexpansive type mapping, Sahar Mohamed Ali proved the existence of a unique fixed point of such mappings when C is closed convex, containing contraction point, and weakly Cauchy subset of a normed space X [12].

Some other generalizations have been given in [11] and [10].

This paper extends and generalizes some of the results given in [2,4,13], and [7] to the $\{a,b,c\}$-*nc*type mappings defined on a closed convex weakly Cauchy subset of a normed space not necessarily Banach in general.

2 Notations and basic definitions

We have the following: