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# On Approximation of Function Classes in Lorentz Spaces with Anisotropic Norm

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**Abstract.** In this paper, Lorentz space of functions of several variables and Besov's class are considered. We establish an exact approximation order of Besov's class by partial sums of Fourier's series for multiple trigonometric system.

**Key Words**: Lorentz space, Besov's class, approximation.

**AMS Subject Classifications**: 42A10, 41A10

## 1 Introduction

Let $\overline{x}=(x_1,\cdots,x_m)\in I^m=[0,2\pi)^m$ and $\theta_j,q_j\in[1,+\infty)$, $j=1,\cdots,m$. We shall denote by $L^*_{\overline{q},\overline{\theta}}(I^m)$ the Lorentz space with anisotropic norm of Lebesgue-measurable functions $f(\overline{x})$ of period $2\pi$ in each variable, such that the quantity

$$\|f\|^*_{\bar{q},\bar{\theta}}=\left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1}\left[\cdots\left[\int_0^{2\pi}\left(f^{*_1,\cdots,*_m}(t_1,\cdots,t_m)\right)^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1}dt_1\right]^{\frac{\theta_2}{\theta_1}}\cdots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right]^{\frac{1}{\theta_m}}$$

is finite, where $f^{*_1,\cdots,*_m}(t_1,\cdots,t_m)$ is the non-increasing rearrangement of the function $|f(\bar{x})|$ in each variable $x_j$ whereas the other variables are fixed (see [7,19]).

Let $\overset{\circ}{L}{}^*_{\overline{q},\overline{\theta}}(I^m)$ be the set of functions $f\in L^*_{\overline{q},\overline{\theta}}(I^m)$, such that

$$\int_0^{2\pi} f(\overline{x})dx_j=0,\quad \forall j=1,\cdots,m,$$

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and let $a_{\overline{n}}(f)$ be the Fourier coefficients of $f\in L_1(I^m)$ with respect to the multiple trigonometric system. Then we set

$$\delta_{\overline{s}}(f,\overline{x})=\sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},$$

where $\langle\bar{y},\bar{x}\rangle=\sum_{j=1}^m y_jx_j$, $\rho(\bar{s})=\{\bar{k}=(k_1,\cdots,k_m)\in\mathbb{Z}^m: 2^{s_j-1}\le|k_j|<2^{s_j},\ j=1,\cdots,m\}$, $\bar{s}=(s_1,\cdots,s_m)\in Z_+^m$.

A number sequence $\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m,}$ belongs to $l_{\overline{p}}$ if

$$\|\{a_{\overline{n}}\}_{\overline{n}\in\mathbb{Z}^m}\|_{l_{\overline{p}}}=\left\{\sum_{n_m=-\infty}^{\infty}\left[\cdots\left[\sum_{n_1=-\infty}^{\infty}|a_{\overline{n}}|^{p_1}\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}\right\}^{\frac{1}{p_m}}<+\infty,$$

where $\overline{p}=(p_1,\cdots,p_m)$, $1\le p_j<+\infty$, $j=1,2,\cdots,m$.

By analogy with [15] and [3] consider the Besov class

$$S^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}B=\left\{f\in \overset{\circ}{L}{}^*_{\overline{p},\overline{\theta}}(I^m)\Big\|\left\{2^{\langle\overline{s},\overline{r}\rangle}\|\delta_{\overline{s}}(f)\|^*_{\overline{p},\overline{\theta}}\right\}\Big\|_{l_{\overline{\tau}}}\le 1\right\},$$

where $\overline{p}=(p_1,\cdots,p_m)$, $\overline{\theta}=(\theta_1,\cdots,\theta_m)$, $\overline{\tau}=(\tau_1,\cdots,\tau_m)$, $1<p_j<+\infty$, $1\le\theta_j,\tau_j<+\infty$, $j=1,\cdots,m$.

For a fixed vector $\bar{\gamma}=(\gamma_1,\cdots,\gamma_m)$, with $\gamma_j>0$, $j=1,\cdots,m$, set

$$Q_n^{\bar{\gamma}}=\cup_{\langle\bar{s},\bar{\gamma}\rangle<n}\rho(\bar{s}),\quad T(Q_n^{\bar{\gamma}})=\Big\{t(\bar{x})=\sum_{\bar{k}\in Q_n^{\bar{\gamma}}} b_{\bar{k}}e^{i\langle\bar{k},\bar{x}\rangle}\Big\}.$$

$E_n^{(\overline{\gamma})}(f)_{\overline{p},\overline{\theta}}$ is the best approximation of a function $f\in L^*_{\overline{p},\overline{\theta}}(I^m)$ by polynomials in $T(Q_n^{\bar{\gamma}})$, and $S_n^{\bar{\gamma}}(f,\bar{x})=\sum_{\bar{k}\in Q_n^{\bar{\gamma}}} a_{\bar{k}}(f)\cdot e^{i\langle\bar{k},\bar{x}\rangle}$ is a partial sum of the Fourier series of $f$.

As pointed out in [28], the difficulty in the theory of approximation of functions of several variables is the choice of the harmonics of the approximating polynomials. The first author suggesting approximation of functions of several variables by polynomials with harmonics in hyperbolic crosses was K. I. Babenko [4]. After that, approximation of various classes of smooth functions by this method was considered by S. A. Telyakovskii [25], B. S. Mityagin [16], Ya. S. Bugrov [8], N. S. Nikol'skaya [18], E. M. Galeev [12], V. N. Temlyakov [26, 27], Din Dung [11], N. N. Pustovoitov [20], E. S. Belinskii [6], B. S. Kashin and V. N. Temlyakov [14], A. R. DeVore, P. Petrushev and V. N. Temlyakov [9], A. S. Romanyuk [21,22].

In particular, A. S. Romanyuk [21] obtained the following result for the Besov classes in Lebesgue spaces with isotropic norm.

**Theorem 1.1** (A. S. Romanyuk)**.** *Assume that* $1\le p<q<+\infty$, $1\le\tau<+\infty$, $1/p-1/q<r_1=\cdots=r_\nu<r_{\nu+1}\le\cdots\le r_m$. $\gamma_j=r_j/r_1$, $j=1,\cdots,m$. *Then*

$$\sup_{f\in S^{\bar{r}}_{p,\tau}B} E_n^{(\overline{\gamma})}(f)_q=E_n^{(\overline{\gamma})}(S^{\bar{r}}_{p,\tau}B)_q\asymp 2^{-n(r_1+\frac{1}{q}-\frac{1}{p})}\cdot n^{(\nu-1)(\frac{1}{q}-\frac{1}{\tau})_+},$$