New Inequalities on $L_{\gamma}$-Spaces

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Abstract. We consider for a fixed $\mu$, the class of polynomials

$$P_{n,\mu,s}(z) := \left\{ P(z) = z^s \left( a_n z^n + \sum_{j=\mu}^{n-s} a_{n-s-j} z^{n-s-j} \right) ; 1 \leq \mu \leq n-s \right\}$$

of degree $n$, having all zeros in $|z| \leq k$, $k \leq 1$, with $s$-fold zeros at the origin. In this paper, we have obtained inequalities in the reverse direction for the above class of polynomials. Besides, extensions of some Turan-type inequalities for the polar derivative of polynomials have been considered.

Key Words: Polynomial, Zygmund inequality, polar derivative.

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1 Introduction

Let $P_n$ be the class of polynomials $P(z)$ of degree at most $n$. For $P \in P_n$, define

$$\|P\|_{\gamma} := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^\gamma \right\}^{1/\gamma}, \quad 1 \leq \gamma < \infty,$$

$$\|P\|_{\infty} := \text{Max}_{|z|=1} |P(z)| \quad \text{and} \quad m := \text{Min}_{|z|=k} |P(z)|.$$

A famous result of S. Bernstein [14] states that if $P \in P_n$, then

$$\|P\|_{\infty} \leq \|P\|_{\infty} \leq n \|P\|_{\infty}. \quad (1.1)$$

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The result is sharp and equality holds in (1.1) for the polynomial $P(z) = az^n$, where $a \neq 0$. Inequality (1.1) was extended to $L_\gamma$-norm by Zygmund [16] who proved that
\[ \| P' \|_{\gamma} \leq n \| P \|_{\gamma}, \quad \gamma \geq 1. \] (1.2)
For the class of polynomials $P \in P_n$ and $P(z)$ having no zero in $|z| < k$, $k \geq 1$, Malik [12] proved
\[ \| P' \|_{\infty} \leq \frac{n}{1+k} \| P \|_{\infty}. \] (1.3)
The result is best possible and equality holds in (1.3) for the polynomial $P(z) = (z+k)^n$.

As a generalization of (1.3) Bidkham and Dewan [6] proved that, if $P \in P_n$ and $P(z)$ having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$,
\[ \text{Max}_{|z|=R} |P'(z)| \leq \frac{n(R+k)^{n-1}}{(1+k)^n} \| P \|_{\infty}. \] (1.4)
The result is best possible and equality holds in (1.4) for the polynomial $P(z) = (z+k)^n$.

On the other hand, it was shown by Turan [15] that if $P(z)$ has all its in $|z| \leq 1$, then
\[ \| P' \|_{\infty} \geq \frac{n}{2} \| P \|_{\infty}. \] (1.5)
The result is best possible and equality holds in (1.5) for every such polynomial having all its zeros on $|z| = 1$.

Inequality (1.5) was refined by Aziz and Dawood [2] by proving under the same hypothesis that
\[ \| P' \|_{\infty} \geq \frac{n}{2} \left\{ \| P \|_{\infty} + m \right\}. \] (1.6)
Again the result is best possible and equality holds in (1.6) for $P(z) = az^n + \beta$, where $|a| = |\beta|$.

As an extension of (1.5), Malik [12] showed that if $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then
\[ \| P' \|_{\infty} \geq \frac{n}{1+k} \| P \|_{\infty}, \] (1.7)
whereas if $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$ with $s$-fold zero at the origin, then Aziz and Shah [5] proved that
\[ \| P' \|_{\infty} \geq \frac{n+sk}{1+k} \| P \|_{\infty}. \] (1.8)
Both the estimates (1.7) and (1.8) are also sharp and equality in (1.7) holds for the polynomial $P(z) = (z+k)^n$ whereas equality in (1.8) holds for the polynomial $P(z) = z^s(z+k)^{n-s}$, $0 \leq s \leq n$. 