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Some Estimates for the Fourier Transform on Rank 1 Symmetric Spaces

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Abstract. Two estimates useful in applications are proved for the Fourier transform in the space $L^2(X)$, where X a symmetric space, as applied to some classes of functions characterized by a generalized modulus of continuity.

Key Words: Fourier transform, generalized continuity modulus, symmetric space.

AMS Subject Classifications: 41A30, 41A17

1 Introduction and preliminaries

In [2], Abilov et al. proved two useful estimates for the Fourier transform classic in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.

In this paper, we prove the analog of Abilov's results see [2] in the Fourier transform on rank 1 symmetric space.

Let $X=G/K$ where G is a connected noncompact semisimple Lie group with finite center and real rank one and K is a maximal compact subgroup. The form of Cartan decomposition is defined by $g=\epsilon+p$, where ϵ is the Lie algebra of K. And $g=\epsilon+a+n$ is Iwasawa decomposition, where a is a maximal abelian subalgebra of p and n is a nilpotent subalgebra of g. The rank one condition is that $dim a=1$. the nilpotent subalgebra n has root space decomposition $n=n_\gamma+n_{2\gamma}$, where γ and 2γ are the positive roots. Let m_γ and $m_{2\gamma}$ be the respective root space dimensions and set $\rho=\frac{1}{2}(m_\gamma+2m_{2\gamma})$. Choose $H_0\in a$ such that $\gamma(H_0)=1$. This allows identifying a with $\mathbb{R}$ by the map $t\in\mathbb{R}\longleftarrow tH_0\in a$, and denote a^* the real dual space of a.

Let $G=NAK$ be the Iwasawa decomposition of the group G, and g, ϵ, a, and n the respective Lie algebras of the groups G, K, A, and N. Denote by M the centralizer of the

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subgroup A in K and put B = k/M. Let dx be a G-invariant measure on X, and let db and dk be the respective normed K-invariant measure on B and K.

The finite Weyl group W acts on a^*. Suppose that $\sum$ is the set of bounded roots ($\sum\subset a^*$), $\sum^+$ is the set of positive bounded roots, and $a^+=\{h\in a;\, \alpha(h)>0 \text{ for } \alpha\in\sum^+\}$ is the positive Weyl chamber. Let $\langle,\rangle$ be the Killing form on the Lie algebra g. For $\lambda\in a^*$ we denote by H_λ the vector in a such that $\lambda(\mathrm{H})=\langle \mathrm{H}_\lambda,\mathrm{H}\rangle$ for all H in a. Let

$$a^*_+=\{\lambda\in a^*,\ \mathrm{H}_\lambda\in a^+\}.$$

The dimension of X is equal to

$$dim\mathrm{X}=m_\gamma+m_{2\gamma}+1.$$

We return to the case in which X = G/K is an arbitrary symmetric space. Given $g\in\mathrm{G}$, denote by $\mathrm{A}(g)\in a$ the unique element satisfying

$$g=n_1\cdot\exp\mathrm{A}(g)\cdot u,$$

where $u\in\mathrm{K}$ and $n_1\in\mathrm{N}$. For $x=g\mathrm{K}\in\mathrm{X}$ and $b=k\mathrm{M}\in\mathrm{B}=\mathrm{K}/\mathrm{M}$, we put

$$\mathrm{A}(x,b)=\mathrm{A}(k^{-1}g).$$

In terms of this decomposition, the invariant measure dx on X has the form

$$dx=\Delta(t)dtdk,$$

where $\Delta(t)=\Delta_{(\alpha,\beta)}(t)=(2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}$, $\alpha=(m_\gamma+m_{2\gamma}-1)/2$ and $\beta=(m_{2\gamma}-1)/2$. The Laplacian on X is denoted L and its radial part is given by

$$\mathrm{L}_r=\frac{d^2}{dt^2}+\frac{\Delta'(t)}{\Delta(t)}\frac{d}{dt}.$$

The spherical function on X is the unique radial solution to the equation

$$\mathrm{L}\phi=-(\lambda^2+\rho^2)\phi.$$

The spherical function is defined by

$$\phi_\lambda(x)=\phi_\lambda^{(\alpha,\beta)}(x)=\int_{\mathrm{B}}e^{(i\lambda+\rho)\mathrm{A}(x,b)}db.$$

Lemma 1.1. *Let $\alpha>\frac{-1}{2}$, $\alpha\geq\beta\geq\frac{-1}{2}$, and let $t_0>0$. Then for $|\eta|\leq\rho$, there exists a positive constant $C_1=C_1(t_0,\alpha,\beta)$ such that*

$$|1-\phi_{\mu+i\eta}(t)|\geq C_1|1-j_\alpha(\mu t)|,$$

where $j_\alpha(t)$ is a normalized Bessel function of the first kind.

Proof. See [3], Lemma 9. □

Lemma 1.2. *The following inequalities are valid for a Jacobi function $\phi_\lambda(t)$ $(\lambda,t\in\mathbb{R}^+)$:*

1. $|\phi_\lambda(t)|\leq 1$.
2. $|1-\phi_\lambda(t)|\leq t^2(\lambda^2+\rho^2)$.

Proof. See [6], Lemmas 3.1-3.2. □

Consider in the Hilbert space $\mathrm{L}^2(\mathrm{X})=\mathrm{L}^2(\mathrm{X},dx)$ with the norm

$$\|f\|=\|f\|_2=\left(\int_{\mathrm{X}}|f(x)|^2dx\right)^{1/2}.$$

The Fourier transform on X was introduced by S. Helgason (see [4,5]), and defined for all $f\in\mathrm{L}^2(\mathrm{X})$ by the formula

$$\widehat{f}(\lambda,b)=\int_{\mathrm{X}}f(x)e^{(-i\lambda+\rho)\mathrm{A}(x,b)}dx,\quad \lambda\in a^*,\quad b\in\mathrm{B}.$$

The inverse the Fourier transform

$$f(x)=\frac{1}{|\mathrm{W}|}\int_{a^*\times\mathrm{B}}\widehat{f}(\lambda,b)e^{(i\lambda+\rho)\mathrm{A}(x,b)}d\mu(\lambda),$$

where $|\mathrm{W}|$ is the order of the Weyl group, and $d\mu(\lambda)=|c(\lambda)|^{-2}d\lambda$ with $d\lambda$ is the element of the Euclidean measure on a^*, and $c(\lambda)$ is the Harish-Chandra function.

The Fourier transform is an isomorphism of the Hilbert space $\mathrm{L}^2(\mathrm{X})$ onto the Hilbert space $\mathrm{L}^2(a^*_+\times\mathrm{B},d\mu(\lambda)db)$.

The Palancherel formula

$$\int_{\mathrm{X}}|f(x)|^2dx=\int_{a^*_+\times\mathrm{B}}|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db.$$

Introduce the translation operator on X. Denote by $d(x,y)$ the distance between $x,y\in\mathrm{X}$ and let

$$\sigma(x,h)=\{y\in\mathrm{X};\ d(x,y)=h\}$$

be the sphere of radius $h>0$ on X centered at x. Let $d\sigma_x(y)$ be the $(dim\mathrm{X}-1)$-dimensional area element of the sphere $\sigma(x,h)$ and let $|\sigma(h)|$ be the area of the whole sphere $\sigma(x,h)$.

Let $f\in\mathrm{L}^2(\mathrm{X})$, the translation operator S_h is defined by

$$(\mathrm{S}_hf)(x)=\frac{1}{|\sigma(h)|}\int_{\sigma(x,h)}f(y)d\sigma_x(y).$$

From [7], we have

$$\widehat{\mathrm{S}_hf}(\lambda,b)=\phi_\lambda(h)\widehat{f}(\lambda,b),\quad h>0. \tag{1.1}$$

Lemma 1.3. *Let* $f(x)\in \mathrm{L}^2(\mathrm{X})$*, then*

$$\widehat{\mathrm{L}f}(\lambda,b)=-(\lambda^2+\rho^2)\widehat{f}(\lambda,b).$$

Proof. See [6], Lemma 1.4. □

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x)=\mathrm{S}_h f(x)-f(x)=(\mathrm{S}_h-\mathrm{I})f(x),$$
$$\Delta_h^k f(x)=\Delta_h(\Delta_h^{k-1}f(x))=(\mathrm{S}_h-\mathrm{I})^k f(x)=\sum_{i=0}^{k}(-1)^{k-1}\tbinom{k}{i}\mathrm{S}_h^i f(x),$$

where $\mathrm{S}_h^0 f(x)=f(x)$, $\mathrm{S}_h^i f(x)=\mathrm{S}_h(\mathrm{S}_h^{i-1}f(x))$, ($i=1,2,\cdots,k$ and $k=1,2,\cdots$), I is a unit operator in $\mathrm{L}^2(\mathrm{X})$, and

$$\Omega_k(f,\delta)=\sup_{0<h\le\delta}\|\Delta_h^k f(x)\|$$

is the k^{th}-order generalized continuity modulus of $f\in\mathrm{L}^2(\mathrm{X})$.

Denote by $\mathrm{W}_{2,\psi}^{m,k}(\mathrm{L})$ the class of functions $f\in\mathrm{L}^2(\mathrm{X})$ such that

$$\Omega_k(\mathrm{L}^m f,\delta)=\mathcal{O}(\psi(\delta^k)),\quad m=0,1,\cdots,$$

and $\psi(t)$ is an arbitrary function defined on $[0,\infty)$.

From formula (1.1), we have

$$\mathrm{S}_h f(x)=\frac{1}{|\mathrm{W}|}\int_{a^*\times\mathrm{B}}\phi_\lambda(h)\widehat{f}(\lambda,b)e^{(i\lambda+\rho)\mathrm{A}(x,b)}d\mu(\lambda)db$$

and

$$f(x)=\frac{1}{|\mathrm{W}|}\int_{a^*\times\mathrm{B}}\widehat{f}(\lambda,b)e^{(i\lambda+\rho)\mathrm{A}(x,b)}d\mu(\lambda)db.$$

By Parseval identity, we obtain

$$\|\mathrm{S}_h f(x)-f(x)\|^2=\int_0^\infty\int_\mathrm{B}|1-\phi_\lambda(h)|^2|\widehat{f}(\lambda,b)|^2d\mu(\lambda)db.$$

To make the formulas concise, we introduce the notation

$$g(\lambda)=\int_\mathrm{B}|\widehat{f}(\lambda,b)|^2db.$$

Using the Lemma 1.3 and it is easy to show tha for $f\in\mathrm{L}^2(\mathrm{X})$

$$\|\Delta_h^k\mathrm{L}^m f(x)\|^2=\int_0^\infty(\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)d\mu(\lambda). \tag{1.2}$$

2 Estimates for the Fourier transform

Theorem 2.1. *For functions $f(x)\in \mathrm{L}^2(\mathrm{X})$ in the class $\mathrm{W}^{m,k}_{2,\psi}(\mathrm{L})$.*

$$\sqrt{\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)}=\mathcal{O}\Big(N^{-2m}\psi\Big(\Big(\frac{c}{N}\Big)^k\Big)\Big),$$

where $m=0,1,\cdots$, $k=1,2,\cdots$,; $c>0$ is a fixed constant, and $\psi(t)$ is any function defined on the interval $[0,\infty)$.

Proof. In the terms of $j_p(x)$ a normalized Bessel function of the first kind we have (see [1])

$$1-j_p(x)=\mathcal{O}(1), \qquad x\geq 1, \tag{2.1a}$$

$$1-j_p(x)=\mathcal{O}(x^2), \qquad 0\leq x\leq 1, \tag{2.1b}$$

$$\sqrt{hx}J_p(hx)=\mathcal{O}(1), \qquad hx\geq 0, \tag{2.1c}$$

where $J_p(x)$ is Bessel function of the first kind, which is related to $j_p(x)$ by the formula

$$j_p(x)=\frac{2^p\Gamma(p+1)}{x^p}J_p(x). \tag{2.2}$$

Let $f\in \mathrm{W}^{m,k}_{2,\psi}(\mathrm{L})$. By Hölder inequality, we have

$$\begin{aligned}
&\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)-\int_{\lambda\geq N} g(\lambda)j_\alpha(\lambda h)d\mu(\lambda)\\
=&\int_{\lambda\geq N}(1-j_\alpha(\lambda h)g(\lambda)d\mu(\lambda)\\
=&\int_{\lambda\geq N}(1-j_\alpha(\lambda h))((g(\lambda))^{1/2})^{2-\frac{1}{k}}((g(\lambda))^{1/2})^{\frac{1}{k}}d\mu(\lambda)\\
\leq&\Big(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\Big)^{\frac{2k-1}{2k}}\Big(\int_{\lambda\geq N}|1-j_\alpha(\lambda h)|^{2k}g(\lambda)d\mu(\lambda)\Big)^{\frac{1}{2k}}\\
\leq&\frac{1}{C_1}\Big(\int_{\lambda\geq N} g(\lambda)|d\mu(\lambda)\Big)^{\frac{2k-1}{2k}}\Big(\int_{\lambda\geq N}|1-\phi_\lambda(h)|^{2k}g(\lambda)d\mu(\lambda)\Big)^{\frac{1}{2k}}\\
=&\frac{1}{C_1}\Big(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\Big)^{\frac{2k-1}{2k}}\Big(\int_{\lambda\geq N}(\lambda^2+\rho^2)^{-2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)(\lambda^2+\rho^2)^{2m}d\mu(\lambda)\Big)^{\frac{1}{2k}}\\
\leq&\frac{1}{C_1}(N^2+\rho^2)^{\frac{-m}{k}}\Big(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\Big)^{\frac{2k-1}{2k}}\Big(\int_{\lambda\geq N}(\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)d\mu(\lambda)\Big)^{\frac{1}{2k}}.
\end{aligned}$$

From the formula (1.2), we have

$$\int_{\lambda\geq N}(\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)d\mu(\lambda)\leq\|\Delta^k_h\mathrm{L}^m f(x)\|^2.$$

Then,

$$\int_{\lambda\geq N} g(\lambda)d\mu(\lambda) \leq \int_{\lambda\geq N} g(\lambda)j_\alpha(\lambda h)d\mu(\lambda) + \frac{1}{C_1}(N^2+\rho^2)^{\frac{-m}{k}}\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k \mathrm{L}^m f(x)\|^{1/k}.$$

Combining this with formulas (2.1c) and (2.2), we have

$$\begin{aligned}&\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\\ =&\mathcal{O}\left(\int_{\lambda\geq N}(\lambda h)^{-\alpha-\frac{1}{2}} g(\lambda)d\mu(\lambda)\right)+N^{\frac{-2m}{k}}\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k \mathrm{L}^m f(x)\|^{1/k}\\ =&\mathcal{O}(Nh)^{-\alpha-\frac{1}{2}}\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)+N^{\frac{-2m}{k}}\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k \mathrm{L}^m f(x)\|^{1/k}.\end{aligned}$$

Or

$$(1-\mathcal{O}(Nh)^{-\alpha-\frac{1}{2}})\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)=\mathcal{O}(N^{\frac{-2m}{k}})\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k \mathrm{L}^m f(x)\|^{1/k}.$$

Setting $h=\frac{c}{N}$, in the last inequality and choosing $c>0$ such that $1-\mathcal{O}(c^{-\alpha-\frac{1}{2}}\geq\frac{1}{2}$, we obtain

$$\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)=\mathcal{O}(N^{\frac{-2m}{k}})\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \psi^{\frac{1}{k}}\left(\left(\frac{c}{N}\right)^k\right).$$

Then

$$\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)=\mathcal{O}\left(N^{-4m}\psi^2\left(\left(\frac{c}{N}\right)^k\right)\right).$$

This completes the proof of theorem. □

Theorem 2.2. *Let $\psi(t)=t^\nu$, where $(\nu>0)$, then*

$$\sqrt{\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)}=\mathcal{O}(N^{-2m-k\nu})\Longleftrightarrow f\in \mathrm{W}^{m,k}_{2,\psi}(\mathrm{L}),$$

$m=0,1,2,\cdots,k=1,2,\cdots,0<\nu<2.$

Proof. We prove sufficiency by Theorem 2.1: let $f\in \mathrm{W}^{m,k}_{2,\psi}(\mathrm{L})$ and $\psi(t)=t^\nu$ then

$$\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{1/2}=\mathcal{O}(N^{-2m-k\nu}).$$

To prove necessity let

$$\left(\int_{\lambda\geq N} g(\lambda)d\mu(\lambda)\right)^{1/2} = \mathcal{O}(N^{-2m-k\nu}).$$

It is easy to show, that there exists a function $f\in \mathrm{L}^2(\mathrm{X})$ such that $\mathrm{L}f\in \mathrm{L}^2(\mathrm{X})$ and

$$\mathrm{L}^m f(x) = \frac{(-1)^m}{|\mathrm{W}|}\int_{a^*\times \mathrm{B}} (\lambda^2+\rho^2)^m \widehat{f}(\lambda,b)e^{(i\lambda+\rho)\mathrm{A}(x,b)}d\mu(\lambda)db.$$

Hence, by the Plancherel equality, we have

$$\|\Delta_h^k \mathrm{L}^m f(x)\|^2 = \int_0^\infty (\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k} g(\lambda)|d\mu(\lambda).$$

This integral is divided into two

$$\int_0^\infty = \int_{0<\lambda<N} + \int_{\lambda\geq N} = \mathrm{I}_1+\mathrm{I}_2,$$

where $N=[h^{-1}]$. We estimate them separately

$$\begin{aligned}
\mathrm{I}_2 =& \int_{\lambda\geq N}(\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)|d\mu(\lambda)\\
=&\mathcal{O}\left(\int_{\lambda\geq N}(\lambda^2+\rho^2)^{2m}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(\sum_{n=N}^{\infty}\int_n^{n+1}(\lambda^2+\rho^2)^{2m}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(\sum_{n=N}^{\infty}((n+1)^2+\rho^2)^{2m}\int_n^{n+1}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(\sum_{n=N}^{\infty}n^{4m}\int_n^{n+1}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(\sum_{n=N}^{\infty}n^{4m}\int_n^{\infty}g(\lambda)d\mu(\lambda)-\sum_{n=N}^{\infty}n^{4m}\int_{n+1}^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(N^{4m}\int_N^{\infty}g(\lambda)d\mu(\lambda)+\sum_{n=N+1}^{\infty}n^{4m}\int_n^{\infty}g(\lambda)d\mu(\lambda)-\sum_{n=N}^{\infty}n^{4m}\int_{n+1}^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(N^{4m}\int_N^{\infty}g(\lambda)d\mu(\lambda)+\sum_{n=N}^{\infty}((n+1)^{4m}-n^{4m})\int_n^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}\left(N^{4m}\int_N^{\infty}g(\lambda)d\mu(\lambda)+\sum_{n=N}^{\infty}n^{4m-1}\int_n^{\infty}g(\lambda)d\mu(\lambda)\right)
\end{aligned}$$

$$\begin{aligned}
&=\mathcal{O}\Big(N^{4m}N^{-4m-2k\nu}+\sum_{n=N}^{\infty}n^{4m-1}n^{-4m-2k\nu}\Big)\\
&=\mathcal{O}(N^{-2k\nu})+\mathcal{O}(N^{-2k\nu})=\mathcal{O}(N^{-2k\nu})=\mathcal{O}(h^{2k\nu}),
\end{aligned}$$

i.e.,

$$\mathrm{I}_2=\mathcal{O}(h^{2k\nu}).$$

Now, we estimate I_1, by (2) in Lemma 1.2, we obtain

$$\begin{aligned}
\mathrm{I}_1=&\int_{0<\lambda<N}(\lambda^2+\rho^2)^{2m}|1-\phi_\lambda(h)|^{2k}g(\lambda)d\mu(\lambda)\\
=&\mathcal{O}(h^{4k})\int_{0<\lambda<N}(\lambda^2+\rho^2)^{2k+2m}g(\lambda)d\mu(\lambda)\\
=&\mathcal{O}(h^{4k})\sum_{n=0}^{N-1}\int_n^{n+1}(\lambda^2+\rho^2)^{2k+2m}g(\lambda)d\mu(\lambda)\\
=&\mathcal{O}(h^{4k})\sum_{n=0}^{N-1}(n+1)^{4m+4k}\int_n^{n+1}g(\lambda)d\mu(\lambda)\\
=&\mathcal{O}(h^{4k})\sum_{n=0}^{N-1}(n+1)^{4m+4k}\left(\int_n^{\infty}g(\lambda)d\mu(\lambda)-\int_{n+1}^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}(h^{4k})\left(\sum_{n=0}^{N-1}(n+1)^{4m+4k}\int_n^{\infty}g(\lambda)d\mu(\lambda)-\sum_{n=0}^{N-1}(n+1)^{4m+4k}\int_{n+1}^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}(h^{4k})\left(1+\sum_{n=0}^{N-1}((n+1)^{4m+4k}-n^{4m+4k})\int_n^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}(h^{4k})\left(1+\sum_{n=1}^{N-1}n^{4m+4k-1}\int_n^{\infty}g(\lambda)d\mu(\lambda)\right)\\
=&\mathcal{O}(h^{4k})\left(1+\sum_{n=1}^{N-1}n^{4m+4k-1}n^{-4m-2k\nu}\right)\\
=&\mathcal{O}(h^{4k})\left(1+N^{4k-2k\nu}\right)=\mathcal{O}(h^{2k\nu}),
\end{aligned}$$

i.e.,

$$\mathrm{I}_1=\mathcal{O}(h^{2k\nu}).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k\mathrm{L}^m f(x)\|=\mathcal{O}(h^{k\nu}),$$

which means that $f\in \mathrm{W}_{2,\psi}^{m,k}(\mathrm{L})$. □

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