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Coefficient Inequalities for p-Valent Functions

T. Panigrahi[1] and R. N. Mohapatra[2,*]

[1] *Department of Mathematics, School of Applied Sciences, KIIT, Bhubaneswar-751024, Odisha, India*
[2] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*

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Abstract. In the present paper, the authors introduce a new subclass of p-valent analytic functions with complex order defined on the open unit disk $\mathbb{U}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$ and obtain coefficient inequalities for the functions in these class. Application of these results for the functions defined by the convolution are also obtained.

Key Words: p-valent function, subordination, coefficient inequalities, convolution.

AMS Subject Classifications: 30C45

1 Introduction and definition

Let $\mathcal{A}_p(p\in\mathbb{N}:=\{1,2,3,\cdots\})$ be the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \tag{1.1}$$

that are regular and p-valent in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C} \text{ and } |z|<1\}.$$

In particular, for $n=1$, we write $\mathcal{A}_1=\mathcal{A}$.

For the functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n,$$

*Corresponding author. *Email addresses:* trailokyap6@gmail.com (T. Panigrahi), ram.mohapatra@ucf.edu (R. N. Mohapatra)

their convolution (or Hadamard product) denoted by $f*g$, is defined by

$$(f*g)(z)=z^p+\sum_{n=p+1}^{\infty}a_nb_nz^n.$$

For two analytic functions f and g, the function f is subordinate to g, written as $f(z)\prec g(z)$ $(z\in\mathbb{U})$, if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ such that$f(z)=g(w(z))$ $(z\in\mathbb{U})$. It follows from this definition that

$$f(z)\prec g(z)\Longrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation (see [9]).

$$f(z)\prec g(z)(z\in\mathbb{U})\Longleftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathbb{U})\subset g(\mathbb{U}).$$

Let $\phi(z)$ be analytic function in $\mathbb{U}$ with $\phi(0)=1$, $\phi'(0)>0$ and $\Re\{\phi(z)\}>0$ which maps the open unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. In [1] Ali et al. defined and introduced the class $S^*_{b,p}(\phi)$ to be the class of function in $f\in\mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{zf'(z)}{pf(z)}-1\right)\prec\phi(z),\quad (z\in\mathbb{U},\quad b\in\mathbb{C}\setminus\{0\}),$$

and the corresponding class $C_{b,p}(\phi)$ of all functions in $f\in\mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right)\prec\phi(z),\quad (z\in\mathbb{U},\quad b\in\mathbb{C}\setminus\{0\}).$$

Further, they also defined and studied the following classes:

$$R_{b,p}(\phi)=\left\{f\in\mathcal{A}_p:1+\frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}-1\right)\prec\phi(z),\ z\in\mathbb{U},\ b\in\mathbb{C}\setminus\{0\}\right\},$$
$$L_p^M(\alpha,\phi)=\left\{\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)}+\frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\phi(z),\ z\in\mathbb{U},\ \alpha\geq 0\right\},$$

and

$$M_p(\alpha,\phi)=\left\{f\in\mathcal{A}_p:\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\prec\phi(z),\ z\in\mathbb{U},\ \alpha\geq 0\right\}.$$

Further, Ramachandran et al. [5] introduced the class $R_{p,b,\alpha,\beta}(\phi)$ to be the class of function in $f\in\mathcal{A}_p$ for which

$$1+\frac{1}{b}\left[(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha}+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}-1\right]\prec\phi(z),\quad (b\in\mathbb{C}\setminus\{0\},\ 0\leq\beta\leq 1,\ \alpha\geq 0).$$

Motivated by the aforementioned works, in this paper we introduce certain subclass of p-valent functions and obtain the sharp coefficient bounds for the functions in these class. Application of these results for the functions defined by the convolution are also obtained.

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathbb{U}$ onto a region in the right half plane and is symmetric with respect to the real axis, $\phi(0)=1$ and $\phi'(0)>0$. A function $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$ if

$$1+\frac{1}{b}\left\{\left[\frac{zf'(z)}{pf(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{pf(z)}+\frac{\lambda}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}-1\right\}\prec[\phi(z)]^{\gamma}, \tag{1.2}$$

where $0\le\alpha,\lambda\le1$; $0<\beta,\gamma\le1$ and b is a non-zero complex number.

Note that, by specializing the parameters p, b, α, β, γ and λ, the class $\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$ reduces to the following classes studied by various earlier researchers.

- $\mathcal{T}^{1}_{p,b,0,1,0}(\phi)=S^{*}_{b,p}(\phi)$ introduced and studied by Ali et al. [1].
- $\mathcal{T}^{1}_{p,1,0,1,\lambda}(\phi)=L^{M}_{p}(\lambda,\phi)$ introduced and studied by Ali et al. [1].
- $\mathcal{T}^{1}_{p,1,\alpha,1-\alpha,1}(\phi)=M_p(\alpha,\phi)$ introduced and studied by Ali et al. [1].
- $\mathcal{T}^{\gamma}_{1,b,\alpha,\beta,\lambda}(\phi)=M^{\gamma}_{b,\alpha,\beta,\lambda}(\phi)$ introduced and studied Reddy and Sharma [6].
- $\mathcal{T}^{1}_{1,b,0,1,0}(\phi)=S^{*}_{b}(\phi)$ and $\mathcal{T}^{1}_{1,b,0,1,1}(\phi)=C_b(\phi)$ defined and studied by Ravichandran et al. [8].
- $\mathcal{T}^{1}_{1,1,\alpha,\beta,1}(\phi)=M_{\alpha,\beta}(\phi)$ defined and studied by Ravichandran et al. [7].
- $\mathcal{T}^{1}_{1,b,0,1,\lambda}(\phi)=\mathcal{T}^{\lambda}_{b}(\phi)$ introduced and studied by Panigrahi and Murugusundaramoorthy [4].
- $\mathcal{T}^{1}_{1,1,0,1,0}(\phi)=S^{*}(\phi)$ and $\mathcal{T}^{1}_{1,1,0,1,1}(\phi)=C(\phi)$ introduced and studied by Ma and Minda [3].

2 Preliminaries

Let Ω be the class of analytic functions of the form

$$w(z)=w_1z+w_2z^2+\cdots$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)|<1$.

In order to derive our main results, we need the following :

Lemma 2.1 (see [1]). *If $w \in \Omega$, then*

$$|w_2 - tw_1^2| \leq \begin{cases} -t, & t \leq -1, \\ 1, & -1 \leq t \leq 1, \\ t, & t \geq 1. \end{cases}$$

When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}$, $(0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if

$$w(z) = -z\frac{\lambda+z}{1+\lambda z}, \quad (0 \leq \lambda \leq 1),$$

or one of its rotations.

Also the sharp upper bound above can be improved as follows.

When $-1 < t < 1$,

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \leq 1, \quad (-1 < t \leq 0),$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leq 1, \quad (0 < t < 1).$$

Lemma 2.2 (see [2]). *If $w \in \Omega$, then for any complex number t,*

$$|w_2 - tw_1^2| \leq \max\{1, |t|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

3 Coefficient bounds

Theorem 3.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n's are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$ then for any complex number μ,*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|b|p^2\gamma B_1}{2[\alpha p + (p+2\lambda)\beta]} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\gamma-1}{2}B_1 - \frac{b\gamma B_1 \sigma_1}{2[\alpha p + (p+\lambda)\beta]^2}\right|\right\}, \tag{3.1}$$

where

$$\sigma_1 = [\alpha p + (p+\lambda)\beta]^2 + 4\mu p^2[\alpha p + (p+2\lambda)\beta] - p^2(1+2p)(\alpha+\beta) - \lambda\beta(\lambda + 4p + 4p^2). \tag{3.2}$$

The result is sharp.

Proof. Let $f(z)\in\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$. By Definition 1.1, there exists an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$ such that

$$1+\frac{1}{b}\left\{\left[\frac{zf'(z)}{pf(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{pf(z)}+\frac{\lambda}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}-1\right\}=[\phi(w(z))]^{\gamma}. \tag{3.3}$$

A computation shows that

$$\frac{zf'(z)}{pf(z)}=1+\frac{a_{p+1}}{p}z+\left(\frac{2}{p}a_{p+2}-\frac{1}{p}a_{p+1}^2\right)z^2+\left(\frac{3}{p}a_{p+3}+\frac{1}{p}a_{p+1}^3-\frac{3}{p}a_{p+1}a_{p+2}\right)z^3+\cdots.$$

Therefore, we have

$$\left[\frac{zf'(z)}{pf(z)}\right]^{\alpha}=1+\frac{\alpha}{p}a_{p+1}z+\left(\frac{2\alpha}{p}a_{p+2}+\frac{\alpha^2-2p\alpha-\alpha}{2p^2}a_{p+1}^2\right)z^2+\cdots, \tag{3.4}$$

and

$$\begin{aligned}&\left[(1-\lambda)\frac{zf'(z)}{pf(z)}+\frac{\lambda}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}\\ =&1+\frac{\beta(\lambda+p)}{p^2}a_{p+1}z+\left\{\frac{2\beta(p+2\lambda)}{p^2}a_{p+2}\right.\\ &\left.+\frac{\beta(\beta-1)(p+\lambda)^2-2\beta p(p^2+2\lambda p+\lambda)}{2p^4}a_{p+1}^2\right\}z^2+\cdots.\end{aligned} \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\begin{aligned}&\left[\frac{zf'(z)}{pf(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{pf(z)}+\frac{\lambda}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}\\ =&1+\frac{\alpha p+(p+\lambda)\beta}{p^2}a_{p+1}z+\frac{2(\alpha p+(p+2\lambda)\beta)}{p^2}a_{p+2}z^2\\ &+\left\{\frac{\alpha(\alpha-2p-1)}{2p^2}+\frac{\beta(\beta-1)(p+\lambda)^2}{2p^4}+\frac{\alpha\beta(p+\lambda)}{p^3}-\frac{\beta(p^2+2\lambda p+\lambda)}{p^3}\right\}a_{p+1}^2z^2+\cdots.\end{aligned} \tag{3.6}$$

Since

$$\phi(w(z))=1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots,$$

therefore, we have

$$[\phi(w(z))]^{\gamma}=1+\gamma B_1w_1z+\left(\gamma B_1w_2+\gamma B_2w_1^2+\frac{\gamma(\gamma-1)}{2}B_1^2w_1^2\right)z^2+\cdots. \tag{3.7}$$

Using (3.6) and (3.7) in (3.3) and then comparing the coefficients of z and z^2 on both sides, we get

$$\frac{\alpha p+(p+\lambda)\beta}{bp^2}a_{p+1}=\gamma B_1w_1,$$

and

$$\frac{2[\alpha p+(p+2\lambda)\beta]}{p^2}a_{p+2}+\Big\{\frac{\alpha(\alpha-2p-1)}{2p^2}+\frac{\beta(\beta-1)(p+\lambda)^2}{2p^4}+\frac{\alpha\beta(p+\lambda)}{p^3}$$
$$-\frac{\beta(p^2+2\lambda p+\lambda)}{p^3}\Big\}a_{p+1}^2$$
$$=b\left\{\gamma B_1 w_2+\gamma B_2 w_1^2+\frac{\gamma(\gamma-1)}{2}B_1^2 w_1^2\right\},$$

which on simplification give

$$a_{p+1}=\frac{bp^2\gamma B_1 w_1}{\alpha p+(p+\lambda)\beta} \tag{3.8}$$

and

$$a_{p+2}=\frac{bp^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}\left[w_2-w_1^2\left\{bp^4\gamma B_1\sigma-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1\right\}\right], \tag{3.9}$$

where

$$\sigma=\Big[\frac{\alpha(\alpha-2p-1)}{p^2}+\frac{\beta(\beta-1)(p+\lambda)^2}{p^4}+2\frac{\alpha\beta(p+\lambda)}{p^3}$$
$$-\frac{2\beta(p^2+2\lambda p+\lambda)}{p^3}\Big]\frac{1}{2[\alpha p+(p+\lambda)\beta]^2}. \tag{3.10}$$

For any complex number μ, we have

$$|a_{p+2}-\mu a_{p+1}^2|=\frac{|b|p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}|w_2-\nu w_1^2|, \tag{3.11}$$

where

$$\nu=\frac{\{[\alpha p+(p+\lambda)\beta]^2+4\mu p^2[\alpha p+(p+2\lambda)\beta]-p^2(1+2p)(\alpha+\beta)-\lambda\beta(\lambda+4p+4p^2)\}b\gamma B_1}{2[\alpha p+(p+\lambda)\beta]^2}$$
$$-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

An application of Lemma 2.2 to (3.11) gives

$$|a_{p+2}-\mu a_{p+1}^2|\le\frac{|b|p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{b\gamma B_1\sigma_1}{2[\alpha p+(p+\lambda)\beta]^2}\right|\right\},$$

where σ_1 is defined in (3.2). This prove the inequality (3.1).

The result is sharp i-e

$$|a_{p+2}-\mu a_{p+1}^2|=\frac{|b|p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}\quad \text{if}\quad w(z)=z^{2\gamma}$$
$$=\frac{|b|p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}\left|\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{b\gamma B_1\sigma_1}{2[\alpha p+(p+\lambda)\beta]^2}\right|\quad \text{if}\quad w(z)=z^{\gamma}.$$

The proof of the Theorem 3.1 is completed. □

Putting $b=\beta=\gamma=1$ and $\alpha=\lambda=0$ in Theorem 3.1 we get the following result.

Corollary 3.1 (see [1]). Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ where B_n's are real with $B_1>0$ and $B_2\geq 0$. If $f(z)$ given by (1.1) belongs to the class $S_p^*(\phi)$, then for any complex number μ,

$$|a_{p+2}-\mu a_{p+1}^2|\leq\frac{pB_1}{2}\max\left\{1,\left|\frac{B_2}{B_1}+(1-2\mu)pB_1\right|\right\}.$$

The result is sharp.

Taking $b=\beta=\gamma=1$ and $\alpha=0$ in Theorem 3.1 we obtain the result for the class $L_p^M(\lambda,\phi)$ as follows:

Corollary 3.2 (see [1]). Let $\phi(z)=1+B_1z+B_2z^2+\cdots$. If the function $f(z)$ given by (1.1) belongs to the class $L_p^M(\lambda,\phi)$, then for any complex number μ, we have

$$|a_{p+2}-\mu a_{p+1}^2|\leq\frac{p^2B_1}{2(p+2\lambda)}\max\left\{1,\left|\frac{p\{p(2\mu-1)(p+2\lambda)-\lambda\}}{(p+\lambda)^2}B_1-\frac{B_2}{B_1}\right|\right\}.$$

The result is sharp.

Letting $b=\lambda=\gamma=1$ and $\beta=1-\alpha$ in Theorem 3.1 we obtain the result for the class $M_p(\alpha,\phi)$ as follows:

Corollary 3.3 (see [1]). Let $\phi(z)=1+B_1z+B_2z^2+\cdots$. If $f\in M_p(\alpha,\phi)$, then for any complex number μ, we have

$$|a_{p+2}-\mu a_{p+1}^2|\leq\frac{p^2B_1}{2(p+2-2\alpha)}\max\left\{1,\left|\frac{\theta_1\mu-\theta_2}{\theta_3}B_1-\frac{B_2}{B_1}\right|\right\},$$

where

$$\theta_1=4p^2(p+2-2\alpha),\quad \theta_2=(1-\alpha)[2p(p+1)^2+\alpha]+2\alpha p^3,\quad \theta_3=2(p+1-\alpha)^2.$$

The result is sharp.

Taking $b=1$ and considering μ as a real number, we prove the following results for the function f in the class $\mathcal{T}^{\gamma}_{p,1,\alpha,\beta,\lambda}(\phi)$. We denote the class $\mathcal{T}^{\gamma}_{p,1,\alpha,\beta,\lambda}(\phi)$ as $\mathcal{T}^{\gamma}_{p,\alpha,\beta,\lambda}(\phi)$.

Theorem 3.2. *Let $\phi(z)=1+B_1z+B_2z^2+\cdots$, where B_n's are real with $B_1>0$ and $B_2\geq 0$. Let $0\leq\alpha,\lambda\leq 1$, $0<\beta,\gamma\leq 1$ and*

$$\delta_1=\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-s_1^2+s_3+s_4\right],$$
$$\delta_2=\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],$$
$$\delta_3=\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],$$

where

$$s_1=\alpha p+(p+\lambda)\beta, \qquad s_2=[\alpha p+(p+2\lambda)\beta]p^2,$$
$$s_3=p^2(1+2p)(\alpha+\beta), \qquad s_4=\lambda\beta(\lambda+4p+4p^2).$$

If $f(z)$ given by (1.1) *belongs to $\mathcal{T}^{\gamma}_{p,\alpha,\beta,\lambda}(\phi)$, then*

$$|a_{p+2}-\mu a_{p+1}^2|\leq\begin{cases}-\dfrac{p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}v, & \mu\leq\delta_1,\\ \dfrac{p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}, & \delta_1\leq\mu\leq\delta_2,\\ \dfrac{p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}v, & \mu\geq\delta_2.\end{cases} \tag{3.12}$$

Further, if $\delta_1\leq\mu\leq\delta_3$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1+v)\frac{s_1^2}{2\gamma B_1 s_2}|a_{p+1}|^2\leq\frac{p^4\gamma B_1}{2s_2}, \tag{3.13}$$

and if $\delta_3\leq\mu\leq\delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1-v)\frac{s_1^2}{2\gamma B_1 s_2}|a_{p+1}|^2\leq\frac{p^4\gamma B_1}{2s_2}, \tag{3.14}$$

where

$$\delta=\frac{s_1^2+4\mu s_2-s_3-s_4}{2s_1^2},\quad v=\delta\gamma B_1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

These results are sharp.

Proof. Let $f\in\mathcal{T}^{\gamma}_{p,\alpha,\beta,\lambda}(\phi)$. Then proceeding as in Theorem 3.1 with $b=1$ and for any real number μ, we obtain

$$|a_{p+2}-\mu a_{p+1}^2|=\frac{p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]}|w_2-vw_1^2|, \tag{3.15}$$

where

$$\begin{aligned}v=&\frac{\{[\alpha p+(p+\lambda)\beta]^2+4\mu p^2[\alpha p+(p+2\lambda)\beta]-p^2(1+2p)(\alpha+\beta)-\lambda\beta(\lambda+4p+4p^2)\}\gamma B_1}{2[\alpha p+(p+\lambda)\beta]^2}\\ &-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1=\frac{s_1^2+4\mu s_2-s_3-s_4}{2s_1^2}\gamma B_1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1=\delta\gamma B_1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1,\end{aligned}$$

where s_1, s_2, s_3, s_4 and δ are defined in the statement of Theorem 3.2.

By application of Lemma 2.1 to the right hand side of relation (3.15) gives the following cases:

Case-1: If $\mu\leq\delta_1$, then

$$\mu\leq\frac{\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-s_1^2+s_3+s_4}{4s_2}.$$

After simplification, we get $v\leq-1$. Therefore,

$$|w_2-vw_1^2|\leq-v.$$

Case-II: If $\delta_1\leq\mu\leq\delta_2$, then

$$\begin{aligned}&\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-s_1^2+s_3+s_4\right]\\&\leq\mu\leq\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],\end{aligned}$$

which on simplification reduces to

$$-1\leq v\leq 1,$$

which implies

$$|w_2-vw_1^2|\leq 1.$$

Case-III: If $\mu\geq\delta_2$, then

$$\mu\geq\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],$$

which implies

$$v\geq 1$$

and

$$|w_2-vw_1^2|\leq v.$$

Thus, the results of (3.12) is established.

Case-IV: Furthermore, if $\delta_1\leq\mu\leq\delta_3$, then

$$\begin{aligned}&\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-s_1^2+s_3+s_4\right]\\&\leq\mu\leq\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],\end{aligned}$$

which implies that $-1<v\le 0$. Hence

$$\begin{aligned}&|w_2-vw_1^2|+(1+v)|w_1|^2\le 1\\ \Longrightarrow&|a_{p+2}-\mu a_{p+1}^2|+(1+v)\frac{[\alpha p+(p+\lambda)\beta]^2}{2p^2\gamma B_1[(\alpha p+p+2\lambda)\beta]}|a_{p+1}|^2\\ \le&\frac{p^4\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]p^2}\\ \Longrightarrow&|a_{p+2}-\mu a_{p+1}^2|+(1+v)\frac{s_1^2}{2\gamma B_1 s_2}|a_{p+1}|^2\le\frac{p^4\gamma B_1}{2s_2}.\end{aligned}$$

Case-V: If $\delta_3\le\mu\le\delta_2$ then

$$\begin{aligned}&\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right]\\ \le&\mu\le\frac{1}{4s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],\end{aligned}$$

which implies $0<v<1$. Hence

$$|w_2-vw_1^2|+(1-v)|w_1|^2\le 1.$$

Now

$$|a_{p+2}-\mu a_{p+1}^2|+(1-v)\frac{s_1^2}{2\gamma B_1 s_2}|a_{p+1}|^2\le\frac{\gamma B_1 p^4}{2s_2}.$$

Sharpness: To prove that the bounds are sharp, we define the function S_{ϕ_n} $(n=2,3,\cdots)$, T_λ and U_λ $(0\le\lambda\le 1)$ as follows.

$$\begin{aligned}&\left[\frac{zS'_{\phi_n}(z)}{pS_{\phi_n}(z)}\right]^\alpha\left[(1-\lambda)\frac{zS'_{\phi_n}(z)}{pS_{\phi_n}(z)}+\frac{\lambda}{p}\left(1+\frac{zS''_{\phi_n}(z)}{S'_{\phi_n}(z)}\right)\right]^\beta\\ =&[\phi(z^{n-1})]^\gamma,\quad (S_{\phi_n}(0)=S'_{\phi_n}(0)-1=0),\\ &\left[\frac{zT'_\lambda(z)}{pT_\lambda(z)}\right]^\alpha\left[(1-\lambda)\frac{zT'_\lambda(z)}{pT_\lambda(z)}+\frac{\lambda}{p}\left(1+\frac{zT''_\lambda(z)}{T'_\lambda(z)}\right)\right]^\beta\\ =&\left[\phi\left(\frac{z(\lambda+z)}{1+\lambda z}\right)\right]^\gamma,\quad (F_\lambda(0)=F'_\lambda(0)-1=0),\\ &\left[\frac{zU'_\lambda(z)}{pU_\lambda(z)}\right]^\alpha\left[(1-\lambda)\frac{zU'_\lambda(z)}{pU_\lambda(z)}+\frac{\lambda}{p}\left(1+\frac{zU''_\lambda(z)}{U'_\lambda(z)}\right)\right]^\beta\\ =&\left[\phi\left(-z\frac{\lambda+z}{1+\lambda z}\right)\right]^\gamma,\quad (U_\lambda(0)=U'_\lambda(0)-1=0).\end{aligned}$$

Clearly, the functions $S_{\phi_n},T_\lambda,U_\lambda\in\mathcal{T}^\gamma_{p,1,\alpha,\beta,\lambda}(\phi)$. If $\mu<\delta_1$ or $\mu>\delta_2$, then the equality holds if and only if f is S_{ϕ_2} or one of its rotations. When $\delta_1<\mu<\delta_2$, then the equality holds if and only if f is S_{ϕ_3} or one of its rotations. If $\mu=\delta_1$, then the equality holds if and only if f is T_λ or one of its rotations. Furthermore, if $\mu=\delta_2$, the equality holds if and only if f is U_λ or one of its rotations. □

Taking $\beta=\gamma=1$ and $\alpha=\lambda=0$ in Theorem 3.2 we get the following:

Corollary 3.4 (see [1])**.** Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ and

$$\delta_1=\frac{B_2-B_1+pB_1^2}{2pB_1^2},\quad \delta_2=\frac{B_2+B_1+pB_1^2}{2pB_1^2},\quad \delta_3=\frac{B_2+pB_1^2}{2pB_1^2}.$$

If $f(z)$ given by (1.1) belongs to $S_p^*(\phi)$, then

$$|a_{p+2}-\mu a_{p+1}^2|\le\begin{cases}\dfrac{p}{2}[B_2+(1-2\mu)pB_1^2], & \mu\le\delta_1,\\ \dfrac{pB_1}{2}, & \delta_1\le\mu\le\delta_2,\\ -\dfrac{p}{2}[B_2+(1-2\mu)pB_1^2], & \mu\ge\delta_2.\end{cases}$$

Further, if $\delta_1\le\mu\le\delta_3$, then

$$|a_{p+2}-\mu a_{p+1}^2|+\frac{1}{2pB_1}\left(1-\frac{B_2}{B_1}+(2\mu-1)pB_1\right)|a_{p+1}|^2\le\frac{pB_1}{2}.$$

If $\delta_3\le\mu\le\delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+\frac{1}{2pB_1}\left(1+\frac{B_2}{B_1}-(2\mu-1)pB_1\right)|a_{p+1}|^2\le\frac{pB_1}{2}.$$

These results are sharp.

Putting $\beta=\gamma=1$ and $\alpha=0$ in Theorem 3.2,we obtain the result for the class $L_p^M(\lambda,\phi)$.

Corollary 3.5. Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ and let

$$\begin{aligned}\delta_1&=\frac{1}{4s_2B_1^2}[2s_1^2(B_2-B_1)+(s_3+s_4-s_1^2)B_1^2],\\ \delta_2&=\frac{1}{4s_2B_1^2}[2s_1^2(B_2+B_1)+(s_3+s_4-s_1^2)B_1^2],\\ \delta_3&=\frac{1}{4s_2B_1^2}[2B_2s_1^2+(s_3+s_4-s_1^2)B_1^2],\end{aligned}$$

where

$$s_1=p+\lambda,\quad s_2=p^2(p+2\lambda),\quad s_3=p^2(1+2p),\quad s_4=\lambda(\lambda+4p+4p^2).$$

If $f(z)$ given by (1.1) belongs to $L_p^M(\lambda,\phi)$, then

$$|a_{p+2}-\mu a_{p+1}^2| \le \begin{cases} -\dfrac{p^2B_1}{2(p+2\lambda)}v, & \mu\le\delta_1,\\ \dfrac{p^2B_1}{2(p+2\lambda)}, & \delta_1\le\mu\le\delta_2,\\ \dfrac{p^2B_1}{2(p+2\lambda)}v, & \mu\ge\delta_2. \end{cases}$$

Further if $\delta_1\le\mu\le\delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1+v)\frac{s_1^2}{2B_1s_2}|a_{p+1}|^2\le\frac{p^4B_1}{2s_2},$$

and if $\delta_3\le\mu\le\delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1-v)\frac{s_1^2}{2B_1s_2}|a_{p+1}|^2\le\frac{p^4B_1}{2s_2},$$

where

$$\delta=\frac{s_1^2+4\mu s_2-s_3-s_4}{2s_1^2},\quad v=\delta B_1-\frac{B_2}{B_1}.$$

These results are sharp.

Taking $\lambda=\gamma=1$ and $\beta=1-\alpha$ in Theorem 3.2 we get:

Corollary 3.6. Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ and let

$$\delta_1=\frac{1}{4s_2B_1^2}[2(B_2-B_1)s_1^2+(s_3+s_4-s_1^2)B_1^2],$$
$$\delta_2=\frac{1}{4s_2B_1^2}[2s_1^2(B_2+B_1)+(s_3+s_4-s_1^2)B_1^2],$$
$$\delta_3=\frac{1}{4s_2B_1^2}[2B_2s_1^2+(s_3+s_4-s_1^2)B_1^2],$$

where

$$s_1=p+1-\alpha,\quad s_2=p^2(p+2-2\alpha),\quad s_3=p^2(1+2p),\quad s_4=(1-\alpha)(1-\alpha+4p+4p^2).$$

If $f(z)$ given by (1.1) belongs to $M_p(\alpha,\phi)$, then

$$|a_{p+2}-\mu a_{p+1}^2| \le \begin{cases} -\dfrac{p^2B_1}{2(p+2-2\alpha)}v, & \mu\le\delta_1,\\ \dfrac{p^2B_1}{2(p+2-2\alpha)}, & \delta_1\le\mu\le\delta_2,\\ \dfrac{p^2B_1}{2(p+2-2\alpha)}v, & \mu\ge\delta_2. \end{cases}$$

Further if $\delta_1 \le \mu \le \delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1+v)\frac{s_1^2}{2B_1 s_2}|a_{p+1}|^2 \le \frac{p^2 B_1}{2(p+2-\alpha)},$$

and if $\delta_3 \le \mu \le \delta_2$, then

$$|a_{p+2}-\mu a_{p+1}^2|+(1-v)\frac{s_1^2}{2B_1 s_2}|a_{p+1}|^2 \le \frac{p^2 B_1}{2(p+2-\alpha)},$$

where

$$\delta=\frac{s_1^2+4\mu s_2-s_3-s_4}{2s_1^2}, \quad v=\delta B_1-\frac{B_2}{B_1}.$$

These results are sharp.

4 Applications to functions defined by convolution

Let $h(z)=z^p+\sum_{n=p+1}^{\infty}h_n z^n$, $(h_n>0)$. For fixed $h\in\mathcal{A}_p$, we define $\mathcal{T}^{\gamma,h}_{p,b,\alpha,\beta,\lambda}(\phi)$ to be the class of all functions $f\in\mathcal{A}_p$ such that $(f*h)(z)\in\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$. Since $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n\in\mathcal{T}^{\gamma,h}_{p,b,\alpha,\beta,\lambda}(\phi)$ if and only if $(f*h)(z)=z^p+\sum_{n=p+1}^{\infty}a_n h_n z^n\in\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$, we obtain the coefficient estimate for the functions in the class $\mathcal{T}^{\gamma,h}_{p,b,\alpha,\beta,\lambda}(\phi)$ from the corresponding estimate for the function in the class $\mathcal{T}^{\gamma}_{p,b,\alpha,\beta,\lambda}(\phi)$. Applying Theorems 3.1 and 3.2 for the function

$$(f*h)(z)=z^p+a_{p+1}h_{p+1}z^{p+1}+a_{p+2}h_{p+2}z^{p+2}+\cdots,$$

we get the following results.

Theorem 4.1. *Let $h(z)=z^p+\sum_{n=p+1}^{\infty}h_n z^n$, $(h_n>0)$ and let the function $\phi(z)$ given by $\phi(z)=1+\sum_{n=1}^{\infty}B_n z^n$ $(B_1>0)$. If $f(z)\in\mathcal{A}_p$ given by* (1.1) *belong to the class $\mathcal{T}^{\gamma,h}_{p,b,\alpha,\beta,\lambda}(\phi)$, then for complex number μ, we have*

$$|a_{p+2}-\mu a_{p+1}^2|\le\frac{|b|p^2\gamma B_1}{2[\alpha p+(p+2\lambda)\beta]h_{p+2}}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-\frac{b\gamma B_1\sigma_1}{2[\alpha p+(p+\lambda)\beta]^2}\frac{h_{p+2}}{h_{p+1}^2}\right|\right\},$$

where σ_1 is given by (3.2). *The result is sharp.*

Putting $\alpha=\lambda=0$ and $\beta=\gamma=1$ in Theorem 4.1 we obtain the result due to Ali et al. [1, Theorem 2].

Taking $b=1$ and μ to be a real number, we obtain the following result for the class $\mathcal{T}^{\gamma,h}_{p,1,\alpha,\beta,\lambda}(\phi,h)$. We denote the class $\mathcal{T}^{\gamma,h}_{p,1,\alpha,\beta,\lambda}(\phi)$ as $\mathcal{T}^{\gamma,h}_{p,\alpha,\beta,\lambda}(\phi)$.

Theorem 4.2. *Let* $\phi(z)=1+B_1z+B_2z^2+\cdots$. *Let* $0\le\alpha,\lambda\le 1$, $0<\beta,\gamma\le 1$ *and*

$$\delta_1=\frac{h_{p+1}^2}{4h_{p+2}s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1-1\right)-s_1^2+s_3+s_4\right],$$
$$\delta_2=\frac{h_{p+1}^2}{4h_{p+2}s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(1+\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],$$
$$\delta_3=\frac{h_{p+1}^2}{4h_{p+2}s_2}\left[\frac{2s_1^2}{\gamma B_1}\left(\frac{B_2}{B_1}+\frac{\gamma-1}{2}B_1\right)-s_1^2+s_3+s_4\right],$$

where

$$s_1=\alpha p+(p+\lambda)\beta, \qquad s_2=[\alpha p+(p+2\lambda)\beta]p^2,$$
$$s_3=p^2(1+2p)(\alpha+\beta), \qquad s_4=\lambda\beta(\lambda+4p+4p^2).$$

If $f(z)$ *given by* (1.1) *belongs to* $\mathcal{T}^{\gamma,h}_{p,\alpha,\beta,\lambda}(\phi)$, *then*

$$|a_{p+2}-\mu a_{p+1}^2|\le\begin{cases}-\dfrac{p^2\gamma B_1}{2h_{p+2}[\alpha p+(p+2\lambda)\beta]}v, & \mu\le\delta_1,\\ \dfrac{p^2\gamma B_1}{2h_{p+2}[\alpha p+(p+2\lambda)\beta]}, & (\delta_1\le\mu\le\delta_2),\\ \dfrac{p^2\gamma B_1}{2h_{p+2}[\alpha p+(p+2\lambda)\beta]}v, & (\mu\ge\delta_2).\end{cases}\tag{4.1}$$

Further, if $\delta_1\le\mu\le\delta_3$, *then*

$$|a_{p+2}-\mu a_{p+1}^2|+\frac{h_{p+1}^2}{h_{p+2}}(1+v)\frac{s_1^2}{2\gamma B_1s_2}|a_{p+1}|^2\le\frac{p^4\gamma B_1}{2h_{p+2}s_2},\tag{4.2}$$

and if $\delta_3\le\mu\le\delta_2$, *then*

$$|a_{p+2}-\mu a_{p+1}^2|+\frac{h_{p+1}^2}{h_{p+2}}(1-v)\frac{s_1^2}{2\gamma B_1s_2}|a_{p+1}|^2\le\frac{p^4\gamma B_1}{2h_{p+2}s_2},\tag{4.3}$$

where

$$\delta=\frac{s_1^2+4\mu s_2\frac{h_{p+2}}{h_{p+1}^2}-s_3-s_4}{2s_1^2},\quad v=\delta\gamma B_1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1.$$

These results are sharp.

Taking $\alpha=\lambda=0$ and $\beta=\gamma=1$ in Theorem 4.2 we get the result of Ali et al. (Corollary 1, [1]).

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References

[1] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput., 187(2) (2007), 35–46.

[2] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20 (1969), 8–12.

[3] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in: Proceeding of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), Int. Press, (1994), 157–169.

[4] T. Panigrahi and G. Murugusundaramoorthy, The Fekete-Szegö coefficient functional for subclass of analytic functions of complex order, Adv. Studies Contemp. Math., 24(1) (2014), 67–75.

[5] C. Ramachandran, S. Sivasubramanian and H. Silverman, Certain coefficient bounds for p-valent functions, Int. J. Math. Math. Sci., 2007, Art ID: 46576, doi: 10.1155/2007/46576.

[6] T. Ram Reddy and R. B. Sharma, Coefficient Inequalities for certain subclassses of analytic functions, Int. J. Math. Anal., (4)(16) (2010), 771–784.

[7] V. Ravichandran, M. Darus, M. Hussain Khan and K. G. Subramanian, Fekete-Szegö inequality for certain class of analytic functions, Australian J. Math. Anal. Appl., 1(2) (2004), 1–7.

[8] V. Ravichandran, Y. Polatoglu, M. Bolcal and A. Sen, Certain subclasses of starlike and convex functions of complex order, Hacettepe J. Math. Stat., 34 (2005), 9–15.

[9] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.