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Generalized Inverse Analysis on the Domain $\Omega(A,A^+)$ in $B(E,F)$

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Abstract. Let $B(E,F)$ be the set of all bounded linear operators from a Banach space E into another Banach space $F,B^+(E,F)$ the set of all double splitting operators in $B(E,F)$ and $GI(A)$ the set of generalized inverses of $A\in B^+(E,F)$. In this paper we introduce an unbounded domain $\Omega(A,A^+)$ in $B(E,F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, and provide a necessary and sufficient condition for $T\in\Omega(A,A^+)$. Then several conditions equivalent to the following property are proved: $B=A^+(I_F+(T-A)A^+)^{-1}$ is the generalized inverse of T with $R(B)=R(A^+)$ and $N(B)=N(A^+)$, for $T\in\Omega(A,A^+)$, where I_F is the identity on F. Also we obtain the smooth (C^∞) diffeomorphism $M_A(A^+,T)$ from $\Omega(A,A^+)$ onto itself with the fixed point A. Let $S=\{T\in\Omega(A,A^+):R(T)\cap N(A^+)=\{0\}\}$, $M(X)=\{T\in B(E,F):TN(X)\subset R(X)\}$ for $X\in B(E,\mathcal{F})\}$, and $\mathcal{F}=\{M(X):\forall X\in B(E,F)\}$. Using the diffeomorphism $M_A(A^+,T)$ we prove the following theorem: S is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X\in S$. The theorem expands the smooth integrability of $\mathcal{F}$ at A from a local neighborhoold at A to the global unbounded domain $\Omega(A,A^+)$. It seems to be useful for developing global analysis and geomatrical method in differential equations.

Key Words: Generalized inverse analysis, smooth diffeomorphism, smooth submanifold.

AMS Subject Classifications: 47B38, 15A29, 58A05

1 Introduction

Let E, F be two Banach spaces, $B(E,F)$ the set of all linear bounded operators from E into $F,B^+(E,F)$ that of all double splitting operators in $B(E,F)$, and $GI(A)$ that of all generalized inverses of A for $A\in B^+(E,F)$. Write $V(A,A^+)=\{T\in B(E,F):\|T-A\|<\|A^+\|^{-1}\}$ for $A\in B^+(E,F)$ and $A^+\in GI(A),C_A(A^+,T)=I_F+(T-A)A^+$ and $D_A(A^+,T)=I_E+A^+(T-A)$, where I_E and I_F denote the identities on F and E, respectivelly. In 1993, Nashed M. Z. and Chen X. indicated in [1] that if $C_A^+(A^+,T)R(T)\subset R(A)$ for $T\in V(A,A^+)$, then the

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following property holds: $B=A^+C_A^{-1}(A^+,T)=D_A^{-1}(A^+,T)$ is the generalized inverse of T with $R(B)=R(A^+)$ and $N(B)=N(A^+)$. This property is essential to the theory of generalized inverse analysis. Then sevaral conditions equivalent to the property are presented (for details see [2–5]). Let T_x be an operator valued map from a topological spaces X into $B(E,F)$. Especially in 1999, the concept of locally fine point of T_x is introduced, see [2]. Thanks to it the complete rank theorem in advanced calculus and the operator rank theorem both are established, see [2,3]. The previous one gives a complete answer to the rank theorem problem presented by Beger M. S in [6], and the latter expands Penrose theorem from the case of matrices to that of operators (see [3,7] and [5]). Many applications of them are given in [5,8,9] and [10]. So far we may say that the generalized inverse analysis is built. In this paper we introduce the unbouded domain $\Omega(A,A^+)$ in $B(E,F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, and show a necessary and sufficient condition for $T\in\Omega(A,A^+)$. Moreover, several conditions equivalent to the above property for $T\in\Omega(A,A^+)$ are given in Theorem 3.1 in the next Section 3. Also we obtain a smooth diffeomorphism from $\Omega(A,A^+)$ onto itself with a fixed point A, i.e., Theorem 4.1 in Section 4 holds. Let $S=\{T\in\Omega(A,A^+):R(T)\cap N(A^+)=\{0\}\}$ for any $A\in B^+(E,F)$ and $A^+\in GI(A)$, $M(X)=\{T\in B(E,F):TN(X)\subset R(X)\}$, and $\mathcal{F}=\{M(X):\forall X\in B(E,F)\}$. Using this smooth diffeomorphism we prove that S is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X\in S$, i.e., Theorem 4.2 in the next Section 4 holds. The theorem expands the smooth integrability of $\mathcal{F}$ at A as indecated in Theorem 4.1 in [6] from a local neighborhood at A to the global unbounded domain $\Omega(A,A^+)$. These seem to be useful for developing the global analysis and geomatrical method in differential equations.

2 The domain $\Omega(A,A^+)$ in $B(E,F)$

Let $B(F)=B(F,F)$ and $B^X(F)$ be the set of all invertible operators in $B(F)$. Write $C_A(A^+,T)=I_F+(T-A)A^{-1}\in B(F)$ for $A\in B^+(E,F)$ and $A^+\in GI(A)$, where I_F denotes the identity on F. We define

$$\Omega(A,A^+)=\{T\in B(E,F):C_A(A^+,T)\in B^X(F)\}$$

for $A\in B^+(E,F)$ and $A^+\in GI(A)$. For abbreviation, write $P_{R(A^+)}^{N(A)}$, $P_{N(A)}^{R(A^+)}$, $P_{R(A)}^{N(A^+)}$ and $P_{N(A^+)}^{R(A)}$ as $P_{R(A^+)}$, $P_{N(A)}$, $P_{R(A)}$ and $P_{N(A^+)}$, respectively, in the sequel. Then $C_A(A^+,T)=P_{N(A^+)}+TA^{-1}$. We have

Theorem 2.1. *T belongs to $\Omega(A,A^+)$ if and only if the following conditions hold:*

$$N(T)\cap R(A^+)=\{0\}$$

and

$$F=R(TA^+)\oplus N(A^+). \tag{2.1}$$

Proof. Assume $T\in\Omega(A,A^+)$. We claim $N(T)\cap R(A^+)=\{0\}$. Let $x\in N(T)$ satisfy $x=A^+y$ for some $y\in R(A)$, then

$$C_A(A^+,T)y=TA^+y=Tx=0,$$

so that $x=A^+y=0$ because of $T\in\Omega(A,A^+)$. This says $N(T)\cap R(A^+)=\{0\}$. Next go to prove that the equality (2.1) holds. We claim that $N(A^+)\cap R(TA^+)=\{0\}$ and $R(TA^+)$ is closed. Let $y_1\in N(A^+)$ satisfy $y_1=TA^+y_0$ for some $y_0\in R(A)$, then

$$C_A(A^+,T)(y_1-y_0)=y_1-TA^+y_0=0,$$

and so, $y_1=0=y_0$ because of $T\in\Omega(A,A^+)$. So $N(A^+)\cap R(TA^+)=\{0\}$. Note $C_A(A^+,T)R(A)=TA^+R(A)=R(TA^+)$. It follows that $R(TA^+)$ is closed for $T\in\Omega(A,A^+)$ and $R(A)$ is closed. Now go to show the equality (2.1). Obviously, it is enough to show $F\subset R(TA^+)\oplus N(A^+)$. Let z belong to F, and $z=C_A(A^+,T)y$ for some $y\in F$. Then

$$z=P_{N(A^+)}y+TA^+P_{R(A)}y\in R(TA^+)\oplus N(A^+).$$

Thus the necessity of the theorem is proved.

Assume that the conclusions of the theorem hold. We claim $N(C_A(A^+,T))=\{0\}$. Obviously

$$0=C_A(A^+,T)y=P_{N(A^+)}y+TA^+P_{R(A)}y \quad \text{for any } y\in N(C_A(A^+,T)).$$

By (2.1), $P_{N(A^+)}y=TA^+P_{R(A)}=0$. By the assumption $N(T)\cap R(TA^+)=\{0\}$, $P_{R(A)}y=0$, so $y=0$. This says $N(C_A(A^+,T))=\{0\}$. Next go to show that $C_A(A^+,T)$ is surjective. By (2.1) there exist $y_0\in N(A^+)$ and $y_1\in R(A)$ such that

$$y=y_0+TA^+y_1 \quad \text{for any } y\in F.$$

Therefore

$$C_A(A^+,T)(y_0+y_1)=y_0+TA^+y_1=y.$$

The proof of the theorem ends. □

Corollary 2.1. If $T\in\Omega(A,A^+)$, then so does xT, $\forall\lambda\neq 0$.

Note $R(\lambda TA^+)=R(TA^+)$ and $N(\lambda T)=N(T)$, $\forall\lambda\neq 0$. The corollary follows from Theorem 2.1.

Corollary 2.2. $\Omega(A,A^+)$ is an unbounded domain in $B(E,F)$.

Proof. By Corollary 2.1, $\Omega(A,A^+)$ is unbounded. Next go to show that $\Omega(A,A^+)$ is an open set in $B(E,F)$. Let T_0 be arbitrary one of $\Omega(A,A^+)$ and $T_*=C_A(A^+,T_0)\in B^X(F)$. Note that $B^X(F)$ is an open set in $B(F)$ and $C_A(A^+,T):B(E,F)\to B(F)$ is continuous. It is obvious that T_0 is an inner point in $\Omega(A,A^+)$. Thus $\Omega(A,A^+)$ is an unbounded domain in $B(E,F)$. □

The following equalities will be often used in the sequel: for $T\in\Omega(A,A^+)$,

$$C_A^{-1}(A^+,T)P_{N(A^+)}=P_{N(A^+)} \quad \text{and} \quad C_A^{-1}(A^+,T)TA^+P_{R(A)}=P_{R(A)}. \tag{2.2}$$

These equalities are immediate from $C_A(A^+,T)P_{N(A^+)}=P_{N(A^+)}$ and $C_A(A^+,T)P_{R(A)}=TA^+P_{R(A)}$ for $T\in\Omega(A,A^+)$.

3 A basic theorem in generalized inverse analysis in $B^+(E,F)$

In this section we will prove a basic theorem in $B^+(E,F)$ generalized inverse analysis, which expands some inportent results in Theorem 1.1 in [6] from $V(A,A^+)$ to the unbounded domain $\Omega(A,A^+)$.

Theorem 3.1. *The following conditions for $T\in\Omega(A,A^+)$ are equivalent:*
(1) $B=A^+C_A^{-1}(A^+,T)$ is the generalized inverse of T with $N(B)=N(A^+)$ and $R(B)=R(A^+)$;
(2) $C_A^{-1}(A^+,T)R(T)\subset R(A)$;
(3) $C_A^{-1}(A^+,T)TN(A)\subset R(A)$;
(4) $R(TA^+)=R(T)$;
(5) $F=R(T)\oplus N(A^+)$;
(6) $R(T)\cap N(A^+)=\{0\}$.

Proof. To show (1) ⇔ (2). Obviously, $R(B)=R(A^+)$ and

$$N(B)=\{y\in F:C_A^{-1}(A^+,T)y\in N(A^+)\}=N(A^+)$$

because of (2.2). While by (2.2)

$$\begin{aligned}BTB-B&=B(TA^+-C_A(A^+,T))C_A^{-1}(A^+,T)\\&=-A^+C_A^{-1}(A^+,T)P_{N(A^+)}C_A^{-1}(A^+,T)\\&=-A^+P_{N(A^+)}C_A^{-1}(A^+,T)=0\end{aligned}$$

and

$$\begin{aligned}TBT-T&=(TA^+-C_A(A^+,T)C_A^{-1}(A^+,T)T\\&=-P_{N(A^+)}C_A^{-1}(A^+,T)T.\end{aligned}$$

Then one concludes (1) ⇔ (2).
To show (2) ⇔ (3). By (2.2)

$$\begin{aligned}C_A^{-1}(A^+,T)Tx&=C_A^{-1}(A^+,T)T(P_{N(A)}x+A^+Ax)\\&=C_A^{-1}(A^+,T)TP_{N(A)}x+C_A^{-1}(A^+,T)TA^+P_{R(A)}Ax\\&=C_A^{-1}(A^+,T)TP_{N(A)}x+Ax\end{aligned}\tag{3.1}$$

for any $x\in E$. Hence (2) $\Leftrightarrow$ (3).

To show (3) $\Leftrightarrow$ (4). Assume that (3) holds. Write $C_A^{-1}(A^+,T)TP_{N(A)}x=y$, $\forall x\in E$, then $y\in R(A)$. By (3.1)

$$Tx=TA^+y+TA^+Ax,$$

and so, one can conclude $R(T)=R(TA^+)$.

Assume $R(T)=R(TA^+)$. Then for any $x\in E$ there exists $y\in R(A)$ such that $Tx=TA^+y$ and so $C_A^{-1}(A^+,T)Tx=y$ because of (2.2). Thus, by (3.1),

$$C_A^{-1}(A^+,T)TP_{N(A)}x=Ax-y\in R(A),\quad \forall x\in E.$$

Hence (4) holds. This shows (3) $\Leftrightarrow$ (4).

To show (4) $\Leftrightarrow$ (5). Assume (4) holds. By (2.1), $F=R(T)\oplus N(A^+)$, i.e., (5) holds. Assume (5) holds. Also by (2.1), for any $y\in R(T)$ there exists $y_0\in R(TA^+)$ and $y_1\in N(A^+)$ such that $y-y_0=y_1\in N(A^+)$; while $R(T)\cap N(A^+)=\{0\}$ because of (5) and so, $y\in R(TA^+)$. This shows $R(T)=R(TA^+)$.

To show (5) $\Leftrightarrow$ (6). Obviously (5) $\Rightarrow$ (6). Assume $R(T)\cap N(A^+)=\{0\}$. By (2.1), for any $y\in R(T)$ there exists $y_0\in R(TA^+)$ and $y_1\in N(A^+)$ such that $y-y_0=y_1\in N(A^+)$. Then by the assumption $R(T)=R(TA^+)$ one concluds that (5) holds. □

4 Some applications

In this section we consder the following two maps:

$$J_A(A^+,T)=TP_{R(A^+)}+C_A(A^+,T)TP_{N(A)},\quad \forall T\in B(E,F),$$

and

$$M_A(A^+,T)=(T-A)P_{R(A^+)}+C_A^{-1}(A^+,T)T,\quad \forall T\in\Omega(A,A^+).$$

Obviously

$$C_A(A^+,J_A(A^+,T))=P_{N(A^+)}+J_A(A^+,T)A^+=P_{N(A^+)}+TA^+=C_A(A^+,T);$$

by (2.2)

$$\begin{aligned}C_A(A^+,M_A(A^+,T))&=P_{N(A^+)}+M_A(A^+,T)A^+\\&=P_{N(A^+)}+TA^+-P_{R(A)}+C_A^{-1}(A^+,T)TA^+\\&=P_{N(A^+)}+TA^+-P_{R(A)}+P_{R(A)}\\&=C_A(A^+,T),\quad \forall T\in\Omega(A,A^+).\end{aligned}$$

So

$$C_A^{-1}(A^+,J_A(A^+,T))=C_A^{-1}(A^+,T)\quad\text{and}\quad C_A^{-1}(A^+,M_A(A^+,T))=C_A^{-1}(A^+,T)\tag{4.1}$$

for any $T\in\Omega(A^+,T)$. Moreover, we have

Theorem 4.1. *$M_A(A^+,T)$ is a smooth diffeomorphism from $\Omega(A,A^+)$ onto ilself with a fixed point A.*

Proof. Obviously, $M_A(A^+,T)$ is smooth (i.e., of C^∞) in $\Omega(A,A^+)$ and $M_A(A^+,A)=A$. By (4.1) and (2.2)

$$\begin{aligned} M_A(A^+,J_A(A^+,T)) &= (J_A(A^+,T)-A)P_{R(A^+)}+C_A^{-1}(A^+,J_A(A^+,T))J_A(A^+,T) \\ &= (T-A)P_{R(A^+)}+C_A^{-1}(A^+,T)J_A(A^+,T) \\ &= (T-A)P_{R(A^+)}+C_A^{-1}(A^+,T)TP_{R(A^+)}+TP_{N(A)} \\ &= T-A+C_A^{-1}(A^+,T)TP_{R(A^+)}=T, \quad \forall T\in\Omega(A,A^+). \end{aligned}$$

Similarly

$$\begin{aligned} J_A(A^+,M_A(A^+,T)) &= M_A(A^+,T)P_{R(A^+)}+C_A(A^+,M_A(A^+,T))M_A(A^+,T)P_{N(A)} \\ &= (T-A)P_{R(A^+)}+C_A^{-1}(A^+,T)TP_{R(A^+)}+C_A(A^+,T)M_A(A^+,T)P_{N(A)} \\ &= TP_{R(A^+)}+TP_{N(A)}=T, \quad \forall T\in\Omega(A,A^+). \end{aligned}$$

The proof ends. □

Let $S=\{T\in\Omega(A,A^+):R(T)\cap N(A^+)=\{0\}\}$. By Theorem 3.1, $X\in S$ has the generalized inverse X^+ with $R(X^+)=R(A^+)$ and $N(X^+)=N(A^+)$. For $X\in S$ consider

$$M_X(X^+,T)=(T-X)P^{N(X)}_{R(A^+)}+C_X^{-1}(X^+,T)T, \quad \forall T\in\Omega(X,X^+).$$

As indicated in the proof of Theorem 4.1,

$$J_X(X^+,T)=TP^{N(T)}_{R(A^+)}+C_X(X^+,T)TP^{R(A^+)}_{N(X)}$$

is its inverse.

By direct computing

$$J'_X(X^+,T)\Delta T=\Delta TP^{N(X)}_{R(A^+)}+P^{R(X)}_{N(A^+)}\Delta TP^{R(A^+)}_{N(X)}+\Delta TX^+TP^{R(A^+)}_{N(X)}+TX^+\Delta TP^{R(A^+)}_{N(X)}$$

for $X\in S$, where $J'_X(X^+,T)$ denotes the Frechet direvative of $J_X(X^+,T)$ at T. Specially, when $T=X$,

$$-J'_X(X^+,X)\Delta T=\Delta T. \tag{4.2}$$

Note that $J_X(X^+,X)=X$ and $M_X(X^+,J_Y(X^+,T))=T$. It follows

$$M'_X(X^+,X)\Delta T=\Delta T. \tag{4.3}$$

Let $M(X)=\{T\in B(E,F):TN(X)\subset R(X)\}$ for $X\in B(E,F)$. Recall that S is said to be tangent to $M(X)$ at $X\in S$ provided

$$M(X)=\{\dot{C}(0):\text{for any } C^1\text{curve } c(t)\subset S \text{ with } c(0)=X\}. \tag{4.4}$$

Evidently

$$T=P_{R(A)}T+P_{N(A^+)}TP_{R(A^+)}+P_{N(A^+)}TP_{N(A)}, \quad \forall T\in B(E,F),$$

and

$$M(A)=\{P_{R(A)}T+P_{N(A^+)}TP_{R(A^+)}:\forall T\in B(E,F)\}.$$

Write $M_*=\{P_{N(A^+)}TP_{N(A)}:\forall T\in B(E,F)\}$. Obviously $B(E,F)=M(A)\oplus M_*$, i.e., $M(A)$ is splitting in $B(E,F)$. We now are in the position to prove the following Theorem.

Theorem 4.2. *S is a smooth submanifold in $B(E,F)$ and tangent to $M(X)$ at any $X\in S$.*

Proof. By Theorem 4.1, $(M_A(A^+,T),\Omega(A,A^+))$ is a smooth coordinate chart of $B(E,F)$ at any $X\in S$. In view of the equivalence of conditions (2) and (6) in Theorem 3.1,

$$M_A(A^+,S)=M(A)\cap\Omega(A,A^+),$$

and so, $M_A(A^+,S)$ is an open set in $M(A)$. This says that S is a smooth submanifold in $B(E,F)$. Now we are going to show that S is tangent to $M(X)$ at any $X\in S$.

Note $N(X^+)=N(A^+)$ and $X\in\Omega(A,A^+)\cap\Omega(X,X^+)$, $\forall X\in S$. One can conclude that there exists a neighborhood U_X at X satisfying $U_X\subset\Omega(A,A^+)\cap\Omega(X,X^+)$, and so

$$S\cap U_X=\{T\in U_X:R(T)\cap N(X^+)=\{0\}\}.$$

Let $C(t)\subset S\cap U_Y$ be any C^1 curve with $C(0)=X$, then by the equivalence of conditions (2) and (6), $D(t)=M_X(X^+,C(t))$ is a C^1 curve contained in $M(X)$ with $D(0)=X$. By (4.3), $\dot{C}(0)=\dot{D}(0)\in M(X)$. This shows $M(X)\supset\{\dot{C}(0)$: for any C^1 curve $C(t)\subset S$ with $C(0)=X\}$.

Let $X+UT=D(T)\subset M(X)\cap U_X$ for any $T\in M(X)$, then $C(t)=J_X(X^+,D(t))\subset S$ is a C^1 curve with $C(0)=X$. By (4.2), $\dot{C}(0)=\dot{D}(0)=T\in M(X)$. Thus S is tangent to $M(X)$ at any $X\in S$. Then proof ends. ☐

The theorem expands Theorem 4.1 in [6] from the case of a local neighborhood at A to that of global domain $\Omega(A,A^+)$. Also we have

Corollary 4.1. 1. If $T\in S$, then $\lambda T\in S$ for $\lambda\neq 0$;
2. $M_A(A^+,\lambda A)=\lambda A$ for $\lambda\neq 0$.

Proof. The part 1 of the corollary is immedaite from Corollary 2.1. Obviously $J_A(A^+,\lambda A)=\lambda A$. Note $C_A^{-1}(A^+,\lambda A)=(P_{N(A^+)}+\lambda P_{R(A)})^{-1}=P_{N(A^+)}+\frac{1}{\lambda}P_{R(A)}$ for $\lambda\neq 0$, then

$$M_A(A^+,\lambda A)=(\lambda-1)AP_{R(A^+)}+(P_{N(A^+)}+\frac{1}{\lambda}P_{R(A)})\lambda A=\lambda A-A+A=\lambda A, \quad \forall\lambda\neq 0.$$

The proof ends. ☐

Our results seem to be useful for developing global analysis, differential topology and geometrical method in differential equations.

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