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On Characterization of Nonuniform Tight Wavelet Frames on Local Fields

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Abstract. In this article, we introduce a notion of nonuniform wavelet frames on local fields of positive characteristic. Furthermore, we gave a complete characterization of tight nonuniform wavelet frames on local fields of positive characteristic via Fourier transform. Our results also hold for the Cantor dyadic group and the Vilenkin groups as they are local fields of positive characteristic.

Key Words: Nonuniform wavelet frame, tight wavelet frame, Fourier transform. local field.

AMS Subject Classifications: 43A70, 11S85, 42C40, 42C15

1 Introduction

Frames in a Hilbert space was originally introduced by Duffin and Schaeffer [6] in the context of non-harmonic Fourier series. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Frames did not seem to generate much interest until the ground-breaking work of Daubechies et al. [3]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (wavelet) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing and medicine. Today more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer [4,5] for an introduction to frame theory and its applications.

Tight wavelet frames are distinct from the orthonormal wavelets because of redundancy. By relinquishing orthonormality and permitting redundancy, the tight wavelet frames turn out to be significantly easier to construct than the orthonormal wavelets. In

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applications, tight wavelet frames provide representations of signals and images where repetition of the representation is favored and the ideal reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets is kept.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. For more details we refer to [1].

The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on K, or more generally, on local fields of positive characteristic. For example, Jiang et al. [9] pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(K)$. Shah and Debnath [10] have constructed tight wavelet frames on local fields of positive characteristic using the extension principles. As far as the construction of wavelet frames on K via Fourier transforms is concerned, Li and Jiang [8] have established a necessary condition and a set of sufficient conditions for the system

$$\Big\{\psi_{j,k} =: q^{j/2}\psi\big(\mathfrak{p}^{-j}x - u(k)\big) : j,k \in \mathbb{N}_0\Big\} \tag{1.1}$$

to be a frame for $L^2(K)$. These studies were continued by Shah and his colleagues in series of papers [11–15].

Motivated and inspired by the above work, we provide the complete characterization of nonuniform tight wavelet frames on local fields of positive characteristic by means of Fourier transform technique. The paper is tailored as follows. In section 2, we discuss some basic facts about local fields of positive characteristic including the notion of nonuniform wavelet frames on local fields of positive characteristic. In section 3, we provide the complete characterization of nonuniform tight wavelet frames on local fields of positive characteristic by using the machinery of Fourier transform.

2 Basic Fourier analysis on local fields

Let K be a field and a topological space. Then K is called a local field if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [7,16]. In the rest of this paper, we use $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a fixed local field. Then there is an integer $q=|\mathfrak{p}^r|$, where $\mathfrak{p}$ is a fixed prime element of K and r is a positive integer, and a norm $|\cdot|$ on K such that for all $x\in K$ we have $|x|>0$ and for each $x\in K\setminus\{0\}$ we get $|x|=q^k$ for some integer k. This norm is non-Archimedean, that is $|x+y|\le\max\{|x|,|y|\}$ for all $x,y\ inK$ and $|x+y|=\max\{|x|,|y|\}$ whenever $|x|\neq|y|$. Let dx be the Haar measure on the locally compact, topological group $(K,+)$. This measure is normalized so that $\int_{\mathfrak{D}}dx=1$, where $\mathfrak{D}=\{x\in K:|x|\le 1\}$ is the ring of integers in K. Define $\mathfrak{B}=\{x\in K:|x|<1\}$. The set $\mathfrak{B}$ is called the prime ideal in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B}=\langle\mathfrak{p}\rangle=\mathfrak{p}\mathfrak{D}$.

Let $\mathfrak{D}^*=\mathfrak{D}\setminus\mathfrak{B}=\{x\in K:|x|=1\}$. Then, it is easy to verify that $\mathfrak{D}^*$ is a group of units in K^* and if $x\neq 0$, then we may write $x=\mathfrak{p}^k x', x'\in\mathfrak{D}^*$. Moreover, each $\mathfrak{B}^k=\mathfrak{p}^k\mathfrak{D}=\left\{x\in K:|x|<q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the fractional ideals of K^+ (see [16]). Let $\mathcal{U}=\{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x\in K$ can be expressed uniquely as $x=\sum_{\ell=k}^{\infty}c_\ell\mathfrak{p}^\ell$ with $c_\ell\in\mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y\in\mathfrak{B}^k$, then $\chi_y(x)=\chi(yx), x\in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n):n\in\mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then it is proved in [19] that the set $\left\{\chi_{u(n)}:n\in\mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n\in\mathbb{N}_0}$. Since $\mathfrak{D}/\mathfrak{B}\cong GF(q)=\Gamma$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$(see [16]). We choose a set $\{1=\epsilon_0,\epsilon_1,\epsilon_2,\cdots,\epsilon_{c-1}\}\subset\mathfrak{D}^*$ such that $\text{span}\{1=\epsilon_0,\epsilon_1,\epsilon_2,\cdots,\epsilon_{c-1}\}\cong GF(q)$. For $n\in\mathbb{N}_0$ such that

$$0\le n<q,\quad n=a_0+a_1p+\cdots+a_{c-1}p^{c-1},\quad 0\le a_k<p\quad\text{and}\quad k=0,1,\cdots,c-1,$$

we define

$$u(n)=(a_0+a_1\epsilon_1+\cdots+a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n=b_0+b_1q+\cdots+b_sq^s$, $n\ge 0$, $0\le b_k<q$, we set

$$u(n)=u(b_0)+\mathfrak{p}^{-1}u(b_1)+\cdots+\mathfrak{p}^{-s}u(b_s).$$

Then, it is easy to verify that (see [16])

$$\{u(k):k\in\mathbb{N}_0\}=\{-u(k):k\in\mathbb{N}_0\},\quad \{u(k)+u(\ell):k\in\mathbb{N}_0\}=\{u(k):k\in\mathbb{N}_0\},\quad\text{for }\ell\in\mathbb{N}_0,$$

and $u(n)=0\Leftrightarrow n=0$. Further, hereafter we will denote $\chi_{u(n)}$ by $\chi_n, n\ge 0$. We also denote the test function space on K by Ω, i.e., each function f in Ω is a finite linear combination of functions of the form $\mathbf{1}_k(x-h)$, $h\in K$, $k\in\mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that Ω is dense in $L^p(K)$, $1\le p<\infty$, and each function in Ω is of compact support and so is its Fourier transform.

The Fourier transform of a function $f\in L^1(K)$ is defined by

$$\hat{f}(\xi)=\int_K f(x)\overline{\chi_\xi(x)}dx.$$

Note that

$$\hat{f}(\xi)=\int_K f(x)\overline{\chi_\xi(x)}dx=\int_K f(x)\chi(-\xi x)dx.$$

The properties of the Fourier transform on the local field K are quite similar to those of the Fourier analysis on the real line [7,16]. In particular, if $f\in L^1(K)\cap L^2(K)$, then $\hat{f}\in L^2(K)$ and

$$\|\hat{f}\|_2=\|f\|_2.$$

The following are the standard definitions of frames in Hilbert spaces.

Definition 2.1. A sequence $\{f_k:k\in\mathbb{Z}\}$ of elements of a Hilbert space $\mathbb{H}$ is called a frame for $\mathbb{H}$ if there exist constants $A,B>0$ such that

$$A\|f\|_2^2\le\sum_{k\in\mathbb{Z}}|\langle f,f_k\rangle|^2\le B\|f\|_2^2$$

holds for every $f\in\mathbb{H}$, and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. A tight frame refers to the case when $A=B$, and a Parseval frame refers to the case when $A=B=1$.

Given an integer $N\ge1$ and an odd integer r with $1\le r\le qN-1$, r and N are relatively prime, we consider the translation set Λ as

$$\Lambda=\left\{0,\frac{u(r)}{N}\right\}+\mathcal{Z}=\left\{\frac{u(r)k}{N}+u(n):n\in\mathbb{N}_0,\ k=0,1\right\}$$

It is easy to verify that Λ is not necessarily a group nor a uniform discrete set, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$.

For a given $\psi\in L^2(K)$, define the nonuniform wavelet (wavelet) system

$$\mathcal{W}(\psi,j,\lambda)=\left\{\psi_{j,\lambda}=:(qN)^{j/2}\psi\big((\mathfrak{p}^{-1}N)^jx-\lambda\big),\ j\in\mathbb{Z},\ \lambda\in\Lambda\right\}. \tag{2.2}$$

On taking Fourier transform, the system (2.2) can be rewritten as

$$\hat{\psi}_{j,\lambda}(\xi)=(qN)^{-j/2}\hat{\psi}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\overline{\chi_\lambda\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}. \tag{2.3}$$

We call the wavelet system $\mathcal{W}(\psi,j,\lambda)$ a nonuniform wavelet(or wavelet) frame for $L^2(K)$, if there exist constants A and $B, 0<A\le B<\infty$ such that for all $f\in L^2(K)$

$$A\|f\|_2^2\le\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2\le B\|f\|_2^2. \tag{2.4}$$

The largest constant A and the smallest constant B satisfying (2.4) are called the lower and upper wavelet frame bound, respectively. A nonuniform wavelet frame is a tight nonuniform wavelet frame if A and B are chosen so that $A=B$ and the nonuniform wavelet frame is called a Parseval nonuniform wavelet frame if $A=B=1$, i.e.,

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|\langle f,\psi_{j,\lambda}\rangle|^2=\|f\|^2,\quad \forall f\in L^2(K), \tag{2.5}$$

and in this case, every function $f\in L^2(K)$ can be written as

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\langle f,\psi_{j,\lambda}\rangle\psi_{j,\lambda}(x).$$

Since Ω is dense in $L^2(K)$ and closed under the Fourier transform, the set

$$\Omega^0=\left\{f\in\Omega:\mathrm{supp}\hat{f}\subset K\backslash\{0\}\right\}$$

is also dense in $L^2(K)$. Therefore, it is sufficient to verify that the system $\mathcal{W}(\psi,j,\lambda)$ given by (2.2) is a frame and tight frame for $L^2(K)$ if (2.4) and (2.5) hold for all $f\in\Omega^0$.

3 Characterization of nonuniform tight wavelet frames on $L^2(K)$

In order to prove the main result to be presented in this section, we need the following lemma whose proof can be found in [16].

Lemma 3.1. *Let $f\in\Omega^0$ and ψ be in $L^2(K)$. If*

$$ess\ sup\left\{\sum_{j\in\mathbb{Z}}|\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)|^2:\xi\in\mathfrak{B}^{-1}\backslash\mathfrak{D}\right\}<\infty,$$

then

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2=\int_K\left|\hat{f}(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\right|^2d\xi+R_\psi(f), \tag{3.1}$$

where

$$\begin{aligned}R_\psi(f)&=\sum_{j\in\mathbb{Z}}\int_K\overline{\hat{f}(\xi)}\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\left\{\sum_{\ell=0}^{qN-1}\hat{f}\left(\xi+(\mathfrak{p}^{-1}N)^ju(\ell)\right)\overline{\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi+u(\ell)\right)}\right\}d\xi\\&=\sum_{j\in\mathbb{Z}}\sum_{\ell=0}^{qN-1}\int_K\overline{\hat{f}(\xi)}\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\hat{f}\left(\xi+(\mathfrak{p}^{-1}N)^ju(\ell)\right)\overline{\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi+u(\ell)\right)}d\xi.\end{aligned} \tag{3.2}$$

Furthermore, the iterated series in (3.2) *is absolutely convergent.*

The L.H.S of (3.1) converges for all $f\in\Omega^0$ if and only if $\sum_{j\in\mathbb{Z}}|\hat\psi\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)|^2$ is locally integrable in $K\backslash\cup_{j\in\mathbb{Z}}E_j^c$, where E_j is the set of regular points of $\left|\psi\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\right|^2$, which means that for each $x\in E_j$, we have

$$(qN)^n\int_{\xi-x\in\mathfrak{C}^n}|\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)|^2d\xi\to|\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)|^2\quad\text{as}\ \ n\to\infty,$$

where $\mathfrak{C}^n=\{x\in K:|x|\le(qN)^{-n}\}$. Then $|E_j^c|=0$. Thus $|\bigcup_{j\in\mathbb{Z}}E_j^c|=0$.

Now we state and prove our main result concerning the characterization of the wavelet system $\mathcal{W}(\psi,j,\lambda)$ given by (2.2) to be tight frame for $L^2(K)$.

Theorem 3.1. *The wavelet system $\mathcal{W}(\psi,j,\lambda)$ given by* (2.2) *is a tight nonuniform wavelet frame for $L^2(K)$ if and only if ψ satisfies*

$$\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\right|^2=1,\quad\textit{for a.e.}\ \ \xi\in\mathfrak{C}^{-1}\backslash\mathfrak{D},\tag{3.3}$$

and

$$\sum_{j\in\mathbb{N}_0}\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\overline{\hat\psi\left((\mathfrak{p}^{-1}N)^{j}(\xi+u(m))\right)}=0,\quad\textit{for a.e.}\ \xi\in\mathfrak{C}^{-1}\backslash\mathfrak{D},\quad 0\le m\le qN-1.\tag{3.4}$$

Proof. Define

$$t_\psi(u(m),\xi)=\sum_{k\in\mathbb{N}_0}\hat\psi\Big((\mathfrak{p}^{-1}N)^{k}\xi\Big)\overline{\hat\psi\left((\mathfrak{p}^{-1}N)^{k}(\xi+u(m))\right)}.$$

Assume $f\in\Omega^0$, then for each $\ell\in\mathbb{N}$, there exists $k\in\mathbb{N}_0$ and a unique $0\le m\le qN-1$ such that $\ell=(qN)^k m$. Thus, by virtue of (2.1) we have that $\{u(\ell)\}_{\ell\in\mathbb{N}}=\{(\mathfrak{p}^{-1}N)^k u(m):k\in\mathbb{N},0\le m\le qN-1\}$. Since the series in (3.2) is absolutely convergent, we can estimate $R_\psi(f)$ as follows:

$$\begin{aligned}
R_\psi(f)=&\sum_{j\in\mathbb{Z}}\int_K\overline{\hat f(\xi)}\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\left\{\sum_{\ell\in\mathbb{N}}\hat f\Big(\xi+(\mathfrak{p}^{-1}N)^{j}u(\ell)\Big)\overline{\hat\psi\left((\mathfrak{p}^{-1}N)^{-j}\xi+u(l)\right)}\right\}d\xi\\
=&\sum_{j\in\mathbb{Z}}\int_K\overline{\hat f(\xi)}\hat\psi\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\left\{\sum_{k\in\mathbb{N}_0}\sum_{m=0}^{qN-1}\hat f\Big(\xi+(\mathfrak{p}^{-1}N)^{j+k}u(m)\Big)\right.\\
&\left.\times\overline{\hat\psi\left((\mathfrak{p}^{-1}N)^{-j}\xi+(\mathfrak{p}^{-1}N)^{k}u(m)\right)}\right\}d\xi\\
=&\int_K\overline{\hat f(\xi)}\left\{\sum_{k\in\mathbb{N}_0}\sum_{m=0}^{qN-1}\sum_{j\in\mathbb{Z}}\hat f\Big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(m)\Big)\hat\psi\Big((\mathfrak{p}^{-1}N)^{j+k}\xi\Big)\right.\\
&\left.\times\overline{\hat\psi\left((\mathfrak{p}^{-1}N)^{-j+k}\xi+(\mathfrak{p}^{-1}N)^{k}u(m)\right)}\right\}d\xi
\end{aligned}$$

$$=\int_K \overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m=0}^{qN-1}\hat{f}\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)\sum_{k\in\mathbb{N}_0}\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{j-k}\xi\Big)\right.$$
$$\left.\times\overline{\hat{\psi}\big((\mathfrak{p}^{-1}N)^k\big((\mathfrak{p}^{-1}N)^{-j}\xi+u(m)\big)\big)}\right\}d\xi$$
$$=\int_K \overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m=0}^{qN-1}\hat{f}\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)t_\psi(u(m),(\mathfrak{p}^{-1}N)^{-j}\xi)\right\}d\xi.$$

Let us collect the results we have obtained: If $\psi\in L^2(K)$ and $f\in\Omega^0$, then

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|\langle f,\psi_{j,\lambda}\rangle|^2\\ =&\int_K|\hat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}|\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)|^2d\xi\\ &+\int_K\overline{\hat{f}(\xi)}\sum_{j\in\mathbb{Z}}\sum_{m=0}^{qN-1}\hat{f}\Big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(m)\Big)t_\psi(u(m),(\mathfrak{p}^{-1}N)^{-j}\xi)d\xi. \end{aligned}\tag{3.5}$$

The last integrand is integrable and so is the first when $\sum_{j\in\mathbb{Z}}\big|\hat{\psi}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\big|^2$ is locally integrable in $K\backslash\cup_{j\in\mathbb{Z}}E_j^c$. Further, Eq. (3.4) implies that

$$t_\psi(u(m),\xi)=0\quad\text{for all}\quad 0\le m\le qN-1.$$

On Combining (3.5) together with (3.3) and (3.4), we obtain

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\big|\langle f,\psi_{j,\lambda}\rangle\big|^2=\|f\|_2^2,\quad\forall f\in\Omega^0.$$

Since Ω^0 is dense in $L^2(K)$, hence the wavelet system $\mathcal{W}(\psi,j,\lambda)$ given by (2.2) is a tight nonuniform wavelet frame for $L^2(K)$.

Conversely, suppose that the system $\mathcal{W}(\psi,j,\lambda)$ given by (2.2) is a tight nonuniform wavelet frame for $L^2(K)$, then we need to show that the two Eqs. (3.3) and (3.4) are satisfied. Since $\{\psi_{j,\lambda}(x):j\in\mathbb{Z},\lambda\in\Lambda\}$ is a tight nonuniform wavelet frame for $L^2(K)$, then we have

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\big|\langle f,\psi_{j,\lambda}\rangle\big|^2=\|f\|_2^2,\quad\forall f\in\Omega^0.\tag{3.6}$$

Since $\sum_{j\in\mathbb{Z}}|\hat{\psi}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)|^2$ is locally integrable in $K\backslash\cup_{j\in\mathbb{Z}}E_j^c$. Therefore, for each $\xi_0\in K\backslash\cup_{j\in\mathbb{Z}}E_j^c$, we consider

$$\hat{f}_1(\xi)=(qN)^{\frac{M}{2}}\mathbf{1}_M(\xi-\xi_0),$$

where $f=f_1$ and $\mathbf{1}_M(\xi-\xi_0)$ is the characteristic function of $\xi_0+\mathfrak{C}^M$. Then, it follows that for $0\le\ell\le qN-1$, $\overline{\hat{f}(\xi)}\hat{f}(\xi+(\mathfrak{p}^{-1}N)^{-j}u(\ell))\equiv 0$, since ξ and $\xi+(\mathfrak{p}^{-1}N)^{-j}u(\ell)$ cannot be in

$\xi_0+\mathfrak{C}^M$ simultaneously and hence, $\|f_1\|_2^2=1$. Furthermore, we have

$$\begin{aligned}1&=\|f_1\|_2^2=\|\hat{f}_1\|_2^2\\&=\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\big|\langle f,\psi_{j,\lambda}\rangle\big|^2\\&=\int_{\xi_0+\mathfrak{C}^M}\sum_{j\in\mathbb{Z}}(qN)^M\Big|\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\Big|^2d\xi+R_\psi(f_1).\end{aligned}$$

By letting $M\to\infty$, we obtain

$$1=\sum_{j\in\mathbb{Z}}\Big|\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi_0\Big)\Big|^2+\lim_{M\to\infty}R_\psi(f_1). \tag{3.7}$$

Now, we proceed to estimate $R_\psi(f_1)$ as:

$$\begin{aligned}R_\psi(f_1)&=\sum_{j\in\mathbb{Z}}\int_K\overline{\hat{f}_1(\xi)}\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\left\{\sum_{\ell\in\mathbb{N}}\hat{f}_1\Big(\xi+(\mathfrak{p}^{-1}N)^ju(\ell)\Big)\overline{\hat{\psi}\left((\mathfrak{p}^{-1}N)^{-j}\xi+u(\ell)\right)}\right\}d\xi,\\\big|R_\psi(f_1)\big|&\le\sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{N}}\int_K\Big|\hat{f}_1(\xi)\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi\Big)\hat{f}_1\Big(\xi+(\mathfrak{p}^{-1}N)^ju(\ell)\Big)\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi+u(\ell)\Big)\Big|d\xi\\&=\sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{N}}(qN)^j\int_K\Big|\hat{f}_1((\mathfrak{p}^{-1}N)^j\xi)\hat{f}_1\Big((\mathfrak{p}^{-1}N)^j(\xi+u(\ell))\Big)\hat{\psi}(\xi)\hat{\psi}(\xi+u(\ell))\Big|d\xi.\end{aligned}$$

Note that

$$\big|\hat{\psi}(\xi)\hat{\psi}(\xi+u(\ell))\big|\le\frac{1}{2}\Big(\big|\hat{\psi}(\xi)\big|^2+\big|\hat{\psi}(\xi+u(\ell))\big|^2\Big).$$

Therefore, we have

$$\big|R_\psi(f_1)\big|\le\sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{N}}(qN)^j\int_K\Big|\hat{f}_1\Big((\mathfrak{p}^{-1}N)^j\xi\Big)\hat{f}_1\Big((\mathfrak{p}^{-1}N)^j(\xi+u(\ell))\Big)\big|\big|\hat{\psi}(\xi)\Big|^2d\xi. \tag{3.8}$$

Since $u(\ell)\neq0,(\ell\in\mathbb{N})$ and $f_1\in\Omega^0$, there exists a constant $J>0$ such that

$$\hat{f}_1\Big((\mathfrak{p}^{-1}N)^jt\Big)\hat{f}_1\Big((\mathfrak{p}^{-1}N)^jt+(\mathfrak{p}^{-1}N)^ju(\ell))\Big)=0,\quad\forall|j|>J.$$

On the other hand, for each $|j|\le J$, there exists a constant L such that

$$\hat{f}_1\Big((\mathfrak{p}^{-1}N)^jt+(\mathfrak{p}^{-1}N)^ju(\ell))\Big)=0,\quad\forall\ell>L.$$

This means that only finite terms of the series on the R.H.S of (3.8) are non-zero. Consequently, there exits a constant C such that

$$\big|R_\psi(f_1)\big|\le C\|\hat{f}_1\|_\infty^2\|\hat{\psi}\|_2^2=C(qN)^m\|\hat{\psi}\|_2^2,$$

which implies

$$\lim_{M\to\infty}\left|R_\psi(f_1)\right|=0.$$

Hence Eq. (3.7) becomes

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\Big((\mathfrak{p}^{-1}N)^{-j}\xi_0\Big)\right|^2=1.$$

Finally, we must show that if (3.6) hold for all $f\in\Omega^0$, then Eq. (3.4) is true. From equalities (3.5), (3.6) and just established equality (3.3), we have

$$\sum_{j\in\mathbb{Z}}\sum_{m=0}^{qN-1}\int_K\overline{\hat{f}(\xi)}\hat{f}\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)t_\psi(u(m),(\mathfrak{p}^{-1}N)^j\xi)d\xi=0,\quad \forall f\in\Omega^0.$$

By invoking polarization identity, we then have

$$\sum_{j\in\mathbb{Z}}\sum_{m=0}^{qN-1}\int_K\overline{\hat{f}(\xi)}\hat{g}\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)t_\psi(u(m),(\mathfrak{p}^{-1}N)^{-j}\xi)d\xi=0,\quad \forall f,g\in\Omega^0. \tag{3.9}$$

Let us fix $m_0\in\{0,1,2,\cdots,qN-1\}$ and $\xi_0\in K\backslash\cup_{j\in\mathbb{Z}}E_j^c$ such that neither $\xi_0\neq 0$ nor $\xi_0+u(m_0)\neq 0$. Setting $f=f_1$ and $g=g_1$ such that

$$\hat{f}_1(\xi)=(qN)^{\frac{M}{2}}\mathbf{1}_M(\xi-\xi_0)\quad\text{and}\quad \hat{g}_1(\xi)=\hat{f}_1(\xi-u(m_0)).$$

Then, we have

$$\hat{f}_1(\xi)\hat{g}_1(\xi+u(m_0))=(qN)^M\mathbf{1}_M(\xi-\xi_0). \tag{3.10}$$

Now, equality (3.9) can be written as

$$0=(qN)^M\int_{\xi_0+\mathfrak{C}^M}t_\psi(u(m_0),\xi)d\xi+I_1,$$

where

$$I_1=\sum_{\substack{j\in\mathbb{Z}\\(j,m)\neq(0,m_0)}}\sum_{m=0}^{qN-1}\int_K\overline{\hat{f}_1(\xi)}\hat{g}_1\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)t_\psi(u(m),(\mathfrak{p}^{-1}N)^{-j}\xi)d\xi. \tag{3.11}$$

Since the first summand in (3.11) tends to $t_\psi(u(m_0),\xi_0)$ as $M\to\infty$. Therefore, we shall prove that

$$\lim_{M\to\infty}I_1=0.$$

Since $u(m)\neq 0$, $(m\in\mathbb{N})$ and $f_1,g_1\in\Omega^0$, there exists a constant $J_0>0$ such that

$$\overline{\hat{f}_1(\xi)}\hat{g}_1\Big(\xi+(\mathfrak{p}^{-1}N)^j u(m)\Big)=0,\quad \forall j>J_0.$$

Therefore, we have

$$I_1=\sum_{j\le J_0}\sum_{m=0}^{qN-1}\int_K\overline{\hat f_1(\xi)}\hat g_1\left(\xi+(\mathfrak{p}^{-1}N)^j u(m)\right)t_\psi(u(m),(\mathfrak{p}^{-1}N)^{-j}\xi)d\xi$$

$$|I_1|\le\sum_{j\le J_0}\sum_{m=0}^{qN-1}(qN)^j\int_K\left|\overline{\hat f_1((\mathfrak{p}^{-1}N)^j\xi)}\hat g_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\right|\left|t_\psi(u(m),\xi)\right|d\xi.$$

Since

$$2\left|t_\psi(u(m),\xi)\right|\le\sum_{k\in\mathbb{N}_0}\left|\hat\psi\left((\mathfrak{p}^{-1}N)^k\xi\right)\right|^2+\sum_{k\in\mathbb{N}_0}\left|\hat\psi\left((\mathfrak{p}^{-1}N)^k(\xi+u(m))\right)\right|^2,$$

hence

$$|I_1|\le I_1^{(1)}+I_1^{(2)},$$

where

$$I_1^{(1)}=\sum_{j\le J_0}\sum_{m=0}^{qN-1}(qN)^j\int_K\left|\hat f_1((\mathfrak{p}^{-1}N)^j\xi)\right|\left|\hat g_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\right|[\tau(\xi)]^2d\xi,$$

with

$$\int_K[\tau(\xi)]^2d\xi=\frac{1}{2}\sum_{k\in\mathbb{N}_0}\int_K\left|\hat\psi\left((\mathfrak{p}^{-1}N)^k\xi\right)\right|^2d\xi=\|\hat\psi\|_2^2<\infty,$$

and

$$\begin{aligned}I_1^{(2)}&=\sum_{j\le J_0}\sum_{m=0}^{qN-1}(qN)^j\int_K\left|\hat f_1((\mathfrak{p}^{-1}N)^j\xi)\right|\left|\hat g_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\right|[\tau(\xi+u(m))]^2d\xi\\&=\sum_{j\le J_0}\sum_{m=0}^{qN-1}(qN)^j\int_K\left|\hat f_1((\mathfrak{p}^{-1}N)^j(\eta-u(m)))\right|\left|\hat g_1\left((\mathfrak{p}^{-1}N)^j\eta\right)\right|[\tau(\eta)]^2d\xi.\end{aligned}$$

Thus $I_1^{(2)}$ has the same form as $I_1^{(1)}$ with the roles of $\hat f_1$ and $\hat g_1$ interchanged. As

$$\hat f_1(\xi)=(qN)^{\frac{M}{2}}\mathbf{1}_M(\xi-\xi_0),$$

therefore, we deduce that

$$I_1^{(1)}=\sum_{j\le J_0}\sum_{m=0}^{qN-1}(qN)^j(qN)^{\frac{M}{2}}\int_{(\mathfrak{p}^{-1}N)^j\xi_0+\mathfrak{C}^{-j+M}}\left|\hat g_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\right|[\tau(\xi)]^2d\xi.$$

Now, if $\hat{g}_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\neq 0$, then we must have $(\mathfrak{p}^{-1}N)^j\xi+(\mathfrak{p}^{-1}N)^j u(m)\in\xi_0+\mathfrak{C}^M+u(m_0)$ and $|(\mathfrak{p}^{-1}N)^j u(m)|\leq (qN)^{-M}$, hence $|u(m)|\leq(qN)^{-M-j}$. Thus,

$$\begin{aligned}
I_1^{(1)}&=\sum_{j\leq J_0}(qN)^j(qN)^{\frac{M}{2}}\int_{(\mathfrak{p}^{-1}N)^j\xi_0+\mathfrak{C}^{-j+M}}[\tau(\xi)]^2\sum_{m=0}^{qN-1}\left|\hat{g}_1\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)\right|d\xi\\
&\leq\sum_{j\leq J_0}(qN)^j(qN)^{\frac{M}{2}}\int_{(\mathfrak{p}^{-1}N)^j\xi_0+\mathfrak{C}^{-j+M}}[\tau(\xi)]^2(qN)^{-M-j}(qN)^{\frac{M}{2}}d\xi\\
&\leq\sum_{j\leq J_0}\int_{(\mathfrak{p}^{-1}N)^j\xi_0+\mathfrak{C}^{-j+M}}[\tau(\xi)]^2d\xi. \qquad (3.12)
\end{aligned}$$

For given $\xi_0\neq 0$, we choose

$$(qN)^{J_0}<|\xi_0|=(qN)^{-M}.$$

Then, we obtain

$$(\mathfrak{p}^{-1}N)^j\xi_0+\mathfrak{C}^{-j+M}\subset\mathfrak{C}^{-J_0+M},\quad\forall j\leq J_0, \qquad (3.13)$$

as $|(\mathfrak{p}^{-1}N)^j\xi_0|=(qN)^j(qN)^{-M}\leq q^{J_0}(qN)^{-M}$ and $\mathfrak{C}^{-j+M}\subset\mathfrak{C}^{-J_0+M}$. On the other hand, for any $j_1<j_2\leq J_0$, we claim that

$$\{(\mathfrak{p}^{-1}N)^{j_1}\xi_0+\mathfrak{C}^{-j_1+M}\}\cap\{(\mathfrak{p}^{-1}N)^{j_2}\xi_0+\mathfrak{C}^{-j_2+M}\}=\emptyset. \qquad (3.14)$$

In fact, for any $x\in(\mathfrak{p}^{-1}N)^{j_1}\xi_0+\mathfrak{C}^{-j_1+M}$ and $y\in(\mathfrak{p}^{-1}N)^{j_2}\xi_0+\mathfrak{C}^{-j_2+M}$, write $x=(\mathfrak{p}^{-1}N)^{j_1}\xi_0+x_1$ and $y=(\mathfrak{p}^{-1}N)^{j_2}\xi_0+y_1$, then

$$|x-y|=\max\{|(\mathfrak{p}^{-1}N)^{j_1}\xi_0-(\mathfrak{p}^{-1}N)^{j_2}\xi_0|,\,|x_1-y_1|\}=(qN)^{j_2-M}\neq 0$$

implies that (3.14) holds. Combining (3.12)-(3.14), we obtain

$$I_1^{(1)}\leq\int_{\mathfrak{C}^{-J_0+M}}[\tau(\xi)]^2d\xi\to 0\quad\text{as}\quad M\to\infty.$$

This completes the proof of the theorem. □

Example 3.1. Let

$$\psi_1(x)=\begin{cases}1, & x\in\mathfrak{D},\\ 0, & x\notin\mathfrak{D},\end{cases}\qquad\text{and}\qquad \psi_2(x)=\begin{cases}(qN)^{-1}, & x\in\mathfrak{C}^{-1},\\ 0, & x\notin\mathfrak{C}^{-1}.\end{cases}$$

Setting $\psi(x)=\psi_1(x)-\psi_2(x)$. Since $\hat{\psi}_1(\xi)=\psi_1(\xi)$ and

$$\hat{\psi}_2(\xi)=\begin{cases}1, & x\in\mathfrak{C},\\ 0, & x\notin\mathfrak{C}.\end{cases}$$

Therefore, we have

$$\hat{\psi}(\xi)=\begin{cases}1, & x\in\mathfrak{C}^{-1}\setminus\mathfrak{D},\\ 0, & \text{otherwise.}\end{cases}$$

Now, for $\xi \neq 0$, we see that

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left((\mathfrak{p}^{-1}N)^j\xi\right)\right|^2 = 1,$$

and since $(\mathfrak{p}^{-1}N)^j\xi$ and $(\mathfrak{p}^{-1}N)^j(\xi+u(m))$ cannot be in $\mathfrak{C}^{-1}\backslash\mathfrak{D}$ simultaneously. Therefore,

$$\sum_{j=0}^{\infty} \hat{\psi}\left((\mathfrak{p}^{-1}N)^j\xi\right)\overline{\hat{\psi}\left((\mathfrak{p}^{-1}N)^j(\xi+u(m))\right)} = 0.$$

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