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Certain Results for the 2-Variable Peters Mixed Type and Related Polynomials

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Abstract. In this article, the 2-variable general polynomials are taken as base with Peters polynomials to introduce a family of 2-variable Peters mixed type polynomials. These polynomials are framed within the context of monomiality principle and their properties are established. Certain summation formulae for these polynomials are also derived. Examples of some members belonging to this family are considered and numbers related to some mixed special polynomials are also explored.

Key Words: 2-Variable general polynomials, Peters polynomials, 2-variable truncated exponential polynomials, Sheffer sequences, monomiality principle.

AMS Subject Classifications: 33C45, 33C50, 33E20

1 Introduction and preliminaries

Special functions (or special polynomials) because of their remarkable properties known as "useful functions" and have been used for centuries. In the past years, the development of new special functions and of applications of special functions to new areas of mathematics have initiated a revival of interest in the p-adic analysis, q-analysis, analytic number theory, combinatorics and so on. Moreover, in recent years, the generalized and multi-variable forms of the special functions of mathematical physics have witnessed a significant evolution. In particular, the special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems see for example [3,4,6,7,11]. Most of the special functions of mathematical physics and their generalizations have been suggested by physical problems. Also these polynomials are helpful in introducing new families of special polynomials.

Motivated by the importance of the special functions of two variables in applications, a general class of the 2-variable polynomials, namely the 2-variable general polynomials

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Table 1: List of special cases of 2VGP $p_n(x,y)$.

S. No.	$\phi(y,t)$	Generating Functions	Polynomials
I.	e^{yt^m}	$\exp(xt+yt^m)=\sum\limits_{n=0}^{\infty} H_n^{(m)}(x,y)\frac{t^n}{n!}$	Gould-Hopper polynomials [10]
II.	e^{yt^2}	$\exp(xt+yt^2)=\sum\limits_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$	2-variable Hermite Kampé de Feriet polynomials [2]
III.	$C_0(-yt^m)$	$e^{xt}C_0(-yt^m)=\sum\limits_{n=0}^{\infty} {}_mL_n(y,x)\frac{t^n}{n!}$	2-variable generalized Laguerre polynomials [7]
IV.	$C_0(-yt)$	$e^{xt}C_0(-yt)=\sum\limits_{n=0}^{\infty} L_n(y,x)\frac{t^n}{n!}$	2-variable Laguerre polynomials [4]
V.	$\frac{1}{(1-yt^r)}$	$\frac{e^{xt}}{(1-yt^r)}=\sum_{n=0}^{\infty} e_n^{(r)}(x,y)t^n$	2-variable truncated exponential polynomials [8] (of order r)
VI.	$\frac{1}{(1-yt^2)}$	$\frac{e^{xt}}{(1-yt^2)}=\sum_{n=0}^{\infty} {}_{[2]}e_n(x,y)t^n$	2-variable truncated exponential polynomials [6]

(2VGP) $p_n(x,y)$ is considered in [11]. These polynomials are defined by the generating function [11, p.4 (14)]

$$e^{xt}\phi(y,t)=\sum_{n=0}^{\infty} p_n(x,y)\frac{t^n}{n!}, \quad p_0(x,y)=1, \tag{1.1}$$

where $\phi(y,t)$ has (at least the formal) series expansion

$$\phi(y,t)=\sum_{n=0}^{\infty}\phi_n(y)\frac{t^n}{n!}, \quad \phi_0(y)\neq 0. \tag{1.2}$$

The 2VGP family contains very important polynomials such as the Gould-Hopper polynomials (GHP) $H_n^{(m)}(x,y)$, 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP) $H_n(x,y)$, 2-variable generalized Laguerre polynomials (2VGLP) ${}_mL_n(y,x)$, 2-variable Laguerre polynomials (2VLP) $L_n(y,x)$, 2-variable truncated exponential polynomials (of order r) (2VTEP) $e_n^{(r)}(x,y)$ and 2-variable truncated exponential polynomials (2VTEP) ${}_{[2]}e_n(x,y)$. We present the list of some known 2VGP family in Table 1.

It is shown in [11], that the polynomials $p_n(x,y)$ are quasi-monomial [4,19] with respect to the following multiplicative and derivative operators:

$$\hat{M}_p=x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)}\Big(\partial_x:=\frac{\partial}{\partial x} \quad \text{and} \quad \phi'(y,t):=\frac{\partial}{\partial t}\phi(y,t)\Big) \tag{1.3}$$

and

$$\hat{P}_p=\partial_x, \tag{1.4}$$

respectively.

According to the monomiality principle and in view of Eqs. (1.3) and (1.4), we have

$$\hat{M}_p\{p_n(x,y)\}=p_{n+1}(x,y) \tag{1.5}$$

and

$$\hat{P}_p\{p_n(x,y)\}=n\ p_{n-1}(x,y), \tag{1.6}$$

respectively.

Now, since the 2VGP $p_n(x,y)$ are quasi-monomial, the properties of these polynomials can be derived from those of the multiplicative and derivative operators $\hat{M}_p$ and $\hat{P}_p$ respectively. In fact, we have

$$\hat{M}_p\hat{P}_p\{p_n(x,y)\}=n\ p_n(x,y), \tag{1.7}$$

which yields the following differential equation satisfied by $p_n(x,y)$:

$$\left(x\partial_x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)}\partial_x-n\right)p_n(x,y)=0. \tag{1.8}$$

Again, since $p_0(x,y)=1$, the 2VGP $p_n(x,y)$ can be explicitly constructed as:

$$p_n(x,y)=\hat{M}_p^n\{p_0(x,y)\}=\hat{M}_p^n\{1\}. \tag{1.9}$$

Identity (1.9) implies that the exponential generating function of the 2VGP $p_n(x,y)$ can be cast in the form

$$\exp(\hat{M}_pt)\{1\}=\sum_{n=0}^{\infty}p_n(x,y)\frac{t^n}{n!},\quad |t|<\infty, \tag{1.10}$$

which yields generating function (1.1). Again, expanding the exponential function e^{xt} and using series expansion (1.2) in the l.h.s. of generating function (1.1), we get the following series definition of the 2VGP $p_n(x,y)$:

$$p_n(x,y)=n!\sum_{k=0}^{n}\frac{x^{n-k}\phi_k(y)}{(n-k)!k!}. \tag{1.11}$$

It can easily be verified that the $\hat{M}_p$ and $\hat{P}_p$ satisfy the following commutation relation:

$$[\hat{P}_p,\hat{M}_p]=\hat{1}. \tag{1.12}$$

Sequences of polynomials play an important role in various branches of sciences. One of the important classes of polynomial sequences is the class of Sheffer sequences. It is a polynomial sequence in which the index of each polynomial equal its degree, satisfying some conditions related to the umbral calculus in combinatorics. There are several ways to define the Sheffer sequences [18], among which by a generating function and by a differential recurrence relation are most common. Sheffer polynomials are classified and many important properties are derived by Rainville [16]. Roman [17] studied these

Sheffer polynomials and handled its properties naturally within the framework of modern and classical umbral calculus. In view of this approach many wonderful results are derived for combinatorics theory, see for example [9,15].

The Peters polynomials (PP) $\mathcal{S}_n(x)$ and their related polynomials Boole polynomials (BlP) $Bl_n(x)$ and the Changhee polynomials (ChP) $Ch_n(x)$, belong to the class of Sheffer sequences. The Peters polynomials $\mathcal{S}_n(x)$ plays an important role in the area of number theory, algebra and umbral calculus and are defined by means of the generating function [12]

$$\sum_{n=0}^{\infty}\mathcal{S}_n(x;\lambda,\mu)\frac{t^n}{n!}=(1+(1+t)^{\lambda})^{-\mu}(1+t)^{x}. \tag{1.13}$$

The first few Peters polynomials are given by [12]

$$\mathcal{S}_0(x;\lambda,\mu)=2^{\mu},\quad \mathcal{S}_1(x;\lambda,\mu)=2^{-(\mu+1)}(2x-\lambda\mu). \tag{1.14}$$

In particular, we note that

$$S_n(x;\lambda;1)=Bl_n(x;\lambda), \tag{1.15a}$$

$$S_n(x;1,1)=Ch_n(x), \tag{1.15b}$$

where $Bl_n(x;\lambda)$ denotes the Boole polynomials (BlP) [13] and $Ch_n(x)$ denotes the Changhee polynomials (ChP) [14] defined by

$$\sum_{n=0}^{\infty}Bl_n(x;\lambda)\frac{t^n}{n!}=(1+(1+t)^{\lambda})^{-1}(1+t)^{x} \tag{1.16}$$

and

$$\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}=\frac{1}{t+2}(1+t)^{x}. \tag{1.17}$$

Taking $x=0$ in generating functions (1.13), (1.16) and (1.17), we find

$$\sum_{n=0}^{\infty}\mathcal{S}_n(\lambda;\mu)\frac{t^n}{n!}=(1+(1+t)^{\lambda})^{-\mu}, \tag{1.18a}$$

$$\sum_{n=0}^{\infty}Bl_n(\lambda)\frac{t^n}{n!}=(1+(1+t)^{\lambda})^{-1}, \tag{1.18b}$$

and

$$\sum_{n=0}^{\infty}Ch_n\frac{t^n}{n!}=\frac{1}{t+2}, \tag{1.19}$$

where

$$\mathcal{S}_n(\lambda;\mu) := \mathcal{S}_n(\lambda;\mu)(0); \quad Bl_n(\lambda) = Bl_n(\lambda)(0); \quad Ch_n := Ch_n(0), \tag{1.20}$$

are the corresponding numbers.

The Stirling number of the first kind is given by

$$(x)_n = x(x-1)(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l. \tag{1.21}$$

Thus, by (1.21), we get

$$(\log(1+t))^m = m! \sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}. \tag{1.22}$$

It is easy to show that

$$x^{(n)} = x(x+1)\cdots(x+n1) = (1)^n(x)_n = \sum_{l=0}^{n} S_1(n,l)(-1)^{n-l}x^l. \tag{1.23}$$

Motivated by the work done on mixed type polynomials and due to the importance of the 2-variable forms of the special polynomials in this paper, a family of the 2-variable Peters mixed type polynomials is introduced by means of generating function and series definition. Certain properties and summation formulae for these polynomials are derived. Examples of some members belonging to this family are considered. The numbers corresponding to certain mixed special polynomials are also explored.

2 2-variable Peters mixed type polynomials

In this section, we introduce the 2-variable Peters mixed type polynomials (2VPmTP) by means of generating function and series definition. Further, we also derive certain properties and summation formulae for these polynomials. In order to derive the generating function for the 2VPmTP, we prove the following result:

Theorem 2.1. *The generating function for the 2-variable Peters mixed type polynomials 2VPmTP* ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ *is given as:*

$$\Big(1+(1+t)^\lambda\Big)^{-\mu} e^{x\ln(1+t)}\phi(y,\ln(1+t)) = \sum_{n=0}^{\infty} {}_p\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{2.1}$$

Proof. Replacing x in the l.h.s. and r.h.s. of generating function (1.13) by the multiplicative operator $\hat{M}_p$ of the 2VGP $p_n(x,y)$, we have

$$\Big(1+(1+t)^\lambda\Big)^{-\mu} \exp\,(\hat{M}_p\ln(1+t))\{1\} = \sum_{n=0}^{\infty} \mathcal{S}_n(\hat{M}_p;\lambda;\mu)\frac{t^n}{n!}, \tag{2.2}$$

Using Eq. (1.10) in the l.h.s. and Eq. (1.3) in the r.h.s. of Eq. (2.2), we find

$$\left(1+(1+t)^{\lambda}\right)^{-\mu}\sum_{n=0}^{\infty}p_n(x,y)\frac{(\ln(1+t))^n}{n!}=\sum_{n=0}^{\infty}\mathcal{S}_n\left(x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)};\lambda;\mu\right)\frac{t^n}{n!}. \tag{2.3}$$

Now, using Eq. (1.1) in the l.h.s. and denoting the resultant 2-variable Peters mixed type polynomials (2VPmTP) in the r.h.s. by ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$, that is

$${}_p\mathcal{S}_n(x,y;\lambda;\mu)=\mathcal{S}_n(\hat{M}_p;\lambda;\mu)=\mathcal{S}_n\left(x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)};\lambda;\mu\right), \tag{2.4}$$

we get assertion (2.1). □

The 2-variable Peters mixed type polynomials (2VPmTP), denoted by ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ will be defined as the discrete Peters convolution of the 2-variable general polynomials $p_n(x,y)$.

Remark 2.1. We remark that Eq. (2.4) gives the operational representation between the Peters polynomials $\mathcal{S}_n(x;\lambda;\mu)$ and 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$.

Next, we obtain the series definition of the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ by proving the following result:

Theorem 2.2. *The 2-variable Peters mixed type polynomials 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ are defined by the series:*

$${}_p\mathcal{S}_n(x,y;\lambda;\mu)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}\mathcal{S}_{n-q}(\lambda;\mu)S_1(q,k)p_k(x,y). \tag{2.5}$$

Proof. Using expansion (1.18a) and Eq. (1.22) in the l.h.s. of Eq. (2.3) and Eq. (2.4) in the r.h.s. of Eq. (2.3), we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=k}^{\infty}\mathcal{S}_n(\lambda;\mu)\ p_k(x,y)\ S_1(q,k)\frac{t^{q+n}}{q!\ n!}=\sum_{n=0}^{\infty}{}_p\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{2.6}$$

Replacing n by $n-q$ in the l.h.s. of Eq. (2.6) and then equating the coefficients of like powers of t in both sides of the resultant equation, we get assertion (2.5). □

Note that from definition (2.5), we conclude that the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ are defined as the discrete Peters convolution of the 2-variable general polynomials $p_n(x,y)$.

Further, to frame the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ within the context of monomiality principle, we prove the following result:

Theorem 2.3. *The 2VPmTP* ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ *are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$\hat{M}_{p\mathcal{S}}=\Big(x+\frac{\phi'(y,e^{\partial_x}-1)}{\phi(y,e^{\partial_x}-1)}-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\Big)\frac{1}{e^{\partial_x}} \tag{2.7}$$

and

$$\hat{P}_{p\mathcal{S}}=e^{\partial_x}-1, \tag{2.8}$$

respectively.

Proof. Consider the identity

$$\partial_x\{e^{x\ln(1+t)}\}=\ln(1+t)\ \{e^{x\ln(1+t)}\}. \tag{2.9}$$

Differentiating Eq. (2.1) partially with respect to t, we find

$$\begin{aligned}&\Big\{\Big(x+\frac{\phi'(y,t)}{\phi(y,t)}-\frac{\lambda\mu(1+t)^\lambda}{1+(1+t)^\lambda}\Big)\frac{1}{(1+t)}\Big\}\Big(1+(1+t)^\lambda\Big)^{-\mu}e^{x\ln(1+t)}\phi(y,\ln(1+t))\\&=\sum_{n=0}^{\infty}{}_p\mathcal{S}_{n+1}(x,y;\lambda;\mu)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Since $\phi(y,t)$ is an invertible series of t, therefore $\frac{\phi'(y,t)}{\phi(y,t)}$ possess power series expansion of t. Thus, in view of the identity (2.9), the above equation becomes

$$\begin{aligned}&\Big\{\Big(x+\frac{\phi'(y,e^{\partial_x}-1)}{\phi(y,e^{\partial_x}-1)}-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\Big)\frac{1}{e^{\partial_x}}\Big\}\Big\{\Big(1+(1+t)^\lambda\Big)^{-\mu}e^{x\ln(1+t)}\phi(y,\ln(1+t))\Big\}\\&=\sum_{n=0}^{\infty}{}_p\mathcal{S}_{n+1}(x,y;\lambda;\mu)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

Now, using generating relation (2.1) in the l.h.s. of Eq. (2.11), rearranging the summation and equating the coefficients of the same powers of t in both sides of the resultant equation, we find

$$\Big\{\Big(x+\frac{\phi'(y,e^{\partial_x}-1)}{\phi(y,e^{\partial_x}-1)}-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\Big)\frac{1}{e^{\partial_x}}\Big\}\Big\{{}_p\mathcal{S}_n(x,y;\lambda;\mu)\Big\}={}_p\mathcal{S}_{n+1}(x,y;\lambda;\mu), \tag{2.12}$$

which in view of monomiality principle equation (1.5) (for ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$) yields assertion (2.7).

Now, to prove assertion (2.8), we note that in view of generating function (2.1) and identity (2.9), we have

$$(e^{\partial_x}-1)\Big\{\sum_{n=0}^{\infty}{}_p\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}\Big\}=\sum_{n=1}^{\infty}{}_p\mathcal{S}_{n-1}(x,y;\lambda;\mu)\frac{t^n}{(n-1)!}. \tag{2.13}$$

Table 2: Special cases of the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$.

S. No.	Values of the parameters	Relation between the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ and its special case	Name of the resultant special polynomials	Generating function and series definition of the resultants special polynomials
I.	$\mu=1$	${}_p\mathcal{S}_n(x,y;\lambda;1)$ $={}_pBl_n(x,y;\lambda)$	2-variable Boole mixed type polynomials (2VBlmTP)	$\Big(1+(1+t)^\lambda\Big)^{-1}e^{x\ln(1+t)}\phi(y,\ln(1+t))=\sum_{n=0}^{\infty}{}_pBl_n(x,y;\lambda)\frac{t^n}{n!}$, ${}_pBl_n(x,y;\lambda)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Bl_{n-q}(\lambda)S_1(q,k)p_k(x,y)$
II.	$\lambda=1,\ \mu=1$	${}_p\mathcal{S}_n(x,y;1;1)$ $={}_pCh_n(x,y)$	2-variable Changhee mixed type polynomials (2VChmTP)	$(t+2)^{-1}e^{x\ln(1+t)}\phi(y,\ln(1+t))=\sum_{n=0}^{\infty}{}_pCh_n(x,y)\frac{t^n}{n!}$, ${}_pCh_n(x,y)=n!\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Ch_{n-q}S_1(q,k)p_k(x,y)$

Rearranging the summation in the l.h.s. of Eq. (2.13) and then equating the coefficients of the same powers of t in both sides of the resultant equation, we find

$$(e^{\partial_x}-1)\Big\{{}_p\mathcal{S}_n(x,y;\lambda;\mu)\Big\}=n\ {}_p\mathcal{S}_{n-1}(x,y;\lambda;\mu),\quad n\geq 1, \tag{2.14}$$

which in view of monomiality principle equation (1.6) (for ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$) yields assertion (2.8). □

We know that the properties of quasi-monomial can be derived by using the expressions of the multiplicative and derivative operators. To derive the differential equation for the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ we prove the following result:

Theorem 2.4. *The 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda,\mu)$ satisfy the following differential equation:*

$$\Big(\frac{x(e^{\partial_x}-1)}{e^{\partial_x}}+\frac{\phi'(y,e^{\partial_x}-1)(e^{\partial_x}-1)}{\phi(y,e^{\partial_x}-1)e^{\partial_x}}-\frac{\lambda\mu e^{(\lambda-1)\partial_x}(e^{\partial_x}-1)}{1+e^{\lambda\partial_x}}\Big)-n\Big){}_p\mathcal{S}_n(x,y;\lambda;\mu)=0. \tag{2.15}$$

Proof. Using expressions (2.7) and (2.8) and in view of monomiality principle equation (1.7), we get assertion (2.15). □

Remark 2.2. We remark that Eqs. (2.12) and (2.14) are the differential recurrence relations satisfied by the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$.

By taking suitable values of the parameters in Eqs. (2.1), (2.4), (2.5), (2.7), (2.8), (2.15) and in view of relations (1.15a) and (1.15b), we can find the generating functions and other results for the mixed special polynomials related to ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$. We present the generating functions and series definitions for these polynomials in Table 2.

It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums involving special functions. The summation formulae of special functions of more than one variable often appear in applications ranging from electromagnetic processes to combinatorics, see for example [5]. The importance of summation formulae of special functions provides motivation to find the summation formulae for the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$.

We derive the summation formulae for the 2VPmTP ${}_p\mathcal{S}_(x,y;\lambda;\mu)$ in the form of following theorems:

Theorem 2.5. *The following implicit summation formula for the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ holds true:*

$$ {}_p\mathcal{S}_n(x+w,y;\lambda;\mu)=\sum_{k=0}^{n}\binom{n}{k}{}_p\mathcal{S}_k(x,y;\lambda;\mu)(w)_{n-k}. \tag{2.16}$$

Proof. Replacing $x\to x+w$ in generating function (2.1), we have

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}e^{(x+w)\ln(1+t)}\phi(y,\ln(1+t))=\sum_{n=0}^{\infty}{}_p\mathcal{S}_n(x+w,y;\lambda;\mu)\frac{t^n}{n!}. \tag{2.17}$$

Again, using Eq. (2.1) and series expansion of e^{wt} in the l.h.s. of Eq. (2.17), we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_p\mathcal{S}_k(x,y;\lambda;\mu)(w)_n\frac{t^{n+k}}{n!k!}=\sum_{n=0}^{\infty}{}_p\mathcal{S}_n(x+w,y;\lambda;\mu)\frac{t^n}{n!}, \tag{2.18}$$

which on replacing n by $n-k$ in the l.h.s. and then equating the coefficients of the same powers of t in both sides of the resultant equation yields assertion (2.16). □

Remark 2.3. We remark that, using Eqs. (1.1), (1.13) and (1.22) in the l.h.s. of Eq. (2.17) and then applying the Cauchy-product rule in the resultant equation, we obtain the following explicit summation formula for the Peters polynomials $\mathcal{S}_n(x;\lambda;\mu)$ in terms of the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$:

$$ {}_p\mathcal{S}_n(x+w,y;\lambda;\mu)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}\mathcal{S}_k(x;\lambda;\mu)S_1(q,k)p_{n-q}(w,y). \tag{2.19}$$

Theorem 2.6. *The following implicit summation formula for the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ holds true:*

$$ {}_p\mathcal{S}_{n+k}(z,y;\lambda;\mu)=\sum_{l,m=0}^{n,k}\frac{n!\;k!\;(z-x)_{l+m}\,{}_p\mathcal{S}_{n+k-l-m}(x,y;\lambda;\mu)}{(n-l)!\;(k-m)!\;l!\;m!}. \tag{2.20}$$

Proof. Replacing $t\to t+u$ in generating function (2.1) and using the following rule:

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{l,m=0}^{\infty}f(l+m)\frac{x^l\;y^m}{l!\;m!}, \tag{2.21}$$

in the r.h.s. of the resultant equation, we find

$$\Big(1+(1+t+u)^{\lambda}\Big)^{-\mu}e^{x\ln(1+t+u)}\phi(y,\ln(1+t+u))=\sum_{n,k=0}^{\infty}{}_p\mathcal{S}_{n+k}(x,y;\lambda;\mu)\frac{t^n\;u^k}{n!\;k!}. \tag{2.22}$$

Replacing x by z in the above equation and then equating the resultant equation to the above equation, we find

$$\sum_{n,k=0}^{\infty} {}_p\mathcal{S}_{n+k}(z,y;\lambda;\mu)\frac{t^n\ u^k}{n!\ k!}=e^{(z-x)\ln(1+t+u)}\sum_{n,k=0}^{\infty} {}_p\mathcal{S}_{n+k}(x,y;\lambda;\mu)\frac{t^n\ u^k}{n!\ k!}, \tag{2.23}$$

which on expanding the exponential in the r.h.s. gives

$$\sum_{n,k=0}^{\infty} {}_p\mathcal{S}_{n+k}(z,y;\lambda;\mu)\frac{t^n\ u^k}{n!\ k!}=\sum_{N=0}^{\infty}\frac{(z-x)_N(t+u)^N}{N!}\sum_{n,k=0}^{\infty} {}_p\mathcal{S}_{n+k}(x,y;\lambda;\mu)\frac{t^n\ u^k}{n!\ k!}. \tag{2.24}$$

Now, using Eq. (2.21) in the r.h.s. of Eq. (2.24) and replacing $n\to n-l$ and $k\to k-m$ in the r.h.s. of the resultant equation, we find

$$\begin{aligned}&\sum_{n,k=0}^{\infty} {}_p\mathcal{S}_{n+k}(z,y;\lambda;\mu)\frac{t^n\ u^k}{n!\ k!}\\ =&\sum_{n,k=0}^{\infty}\sum_{l,m=0}^{n,k}\frac{(z-x)_{l+m}}{l!\ m!}{}_p\mathcal{S}_{n+k-l-m}(x,y;\lambda;\mu)\frac{t^n\ u^k}{(n-l)!\ (k-m)!}.\end{aligned} \tag{2.25}$$

Finally, on equating the coefficients of the same powers of t and u in Eq. (2.25), we are led to assertion (2.20). In the next section, examples of some members belonging to the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ are considered. □

3 Examples

The 2VGP family $p_n(x,y)$ which are classified as an important general class of special functions due to wide range of applications contains a number of important special polynomials of two variables. Certain members belonging to the 2VGP family $p_n(x,y)$ are considered in Section 1. We note that corresponding to each member belonging to the 2VGP $p_n(x,y)$, there exists a new special polynomial belonging to the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ family. Thus, by making suitable choice for the function $\phi(y,t)$ in Eq. (2.1), we get the generating function for the corresponding member belonging to the 2VPmTP ${}_p\mathcal{S}_n(x,y;\lambda;\mu)$ family. The other properties of these special polynomials can be obtained from the results derived in the previous section.

We consider the following examples:

Example 3.1. Taking $\phi(y,t)=e^{yt^m}$ (for which the 2VGP $p_n(x,y)$ reduce to the GHP $H_n^{(m)}(x,y)$ Table 1(I)) in the l.h.s. of generating function (2.1), we find that the resultant Gould-Hopper Peter polynomials (GHPP), denoted by ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$ in the r.h.s. are defined by the following generating function:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}e^{x\ln(1+t)+y(\ln(1+t))^m}=\sum_{n=0}^{\infty}{}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{3.1}$$

The series definitions and other results for the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$ are given in Table 3.

Remark 3.1. Since for $m=2$, the GHP $H_n^{(m)}(x,y)$ reduce to the 2VHKdFP $H_n(x,y)$ (Table 1(II)). Therefore, taking $m=2$ in Eq. (3.1), we get the following generating function for the 2-variable Hermite Peters polynomials (2VHPP), denoted by ${}_H\mathcal{S}_n(x,y;\lambda;\mu)$:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu} e^{x\ln(1+t)+y(\ln(1+t))^2}=\sum_{n=0}^{\infty} {}_H\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{3.2}$$

The series definitions and other results for the 2VHPP ${}_H\mathcal{S}_n(x,y;\lambda;\mu)$ can be obtained by taking $m=2$ in the results given in Table 3.

Remark 3.2. Since for $x\to 2x$ and $y=-1$ the 2VHKdFP $H_n(x,y)$ reduce to the classical Hermite polynomials $H_n(x)$ [1]. Therefore, taking $x\to 2x$ and $y=-1$ in Eq. (3.2), we get the following generating function for the Hermite Peters polynomials (HPP), denoted by ${}_H\mathcal{S}_n(x;\lambda;\mu)$:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu} e^{2x\ln(1+t)-(\ln(1+t))^2}=\sum_{n=0}^{\infty} {}_H\mathcal{S}_n(x;\lambda;\mu)\frac{t^n}{n!}. \tag{3.3}$$

The series definitions and other results for the HPP ${}_H\mathcal{S}_n(x;\lambda;\mu)$ can be obtained by taking $m=2$, $x\to 2x$ and $y=-1$ in the results given in Table 3.

Taking suitable values of the parameters in the results of the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$ and in view of Eqs. (1.15a) and (1.15b), we can find the corresponding results for the mixed special polynomials related to ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$. We use the suitable notations for

Table 3: Results for ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$.

S.No.	Results	Expressions
I.	**Series definition**	${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}\mathcal{S}_{n-q}(\lambda;\mu)S_1(q,k)H_k^{(m)}(x,y)$
II.	**Multiplicative and derivative operators**	$\hat{M}_{{}_{H^{(m)}}\mathcal{S}}=\Big(x+my\partial_x^{m-1}-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\Big)\frac{1}{e^{\partial_x}}$ $\hat{P}_{{}_{H^{(m)}}\mathcal{S}}=e^{\partial_x}-1$
III.	**Differential equation**	$\Big(\frac{x(e^{\partial_x}-1)}{e^{\partial_x}}+\frac{my\partial_x^{m-1}(e^{\partial_x}-1)}{e^{\partial_x}}-\frac{\lambda\mu e^{(\lambda-1)\partial_x}(e^{\partial_x}-1)}{1+e^{\lambda\partial_x}}\Big)-n\Big)_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)=0$
IV.	**Operational rule**	${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)=\mathcal{S}_n(\hat{M}_{H^{(m)}};\lambda;\mu)$; $\hat{M}_{H^{(m)}}$:=Multiplicative operator of $H_n^{(m)}(x,y)$
V.	**Summation formulae**	${}_{H^{(m)}}\mathcal{S}_n(x+w,y;\lambda;\mu)=\sum\limits_{k=0}^{n}\binom{n}{k}{}_{H^{(m)}}\mathcal{S}_k(x,y;\lambda;\mu)(w)_{n-k}$ ${}_{H^{(m)}}\mathcal{S}_{n+k}(z,y;\lambda;\mu)=\sum\limits_{l,m=0}^{n,k}\frac{n!\ k!\ (z-x)_{l+m}\ {}_{H^{(m)}}\mathcal{S}_{n+k-l-m}(x,y;\lambda;\mu)}{(n-l)!\ (k-m)!\ l!\ m!}$ ${}_{H^{(m)}}\mathcal{S}_n(x+w,y;\lambda;\mu)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}\mathcal{S}_k(x;\lambda;\mu)S_1(q,k)H_{n-q}^{(m)}(w,y)$

Table 4: Special cases of the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$.

S. No.	Values of the parameters	Relation between the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu,\nu)$ and its special case	Name of the resultant special polynomials	Generating function and series definition of the resultant special polynomials
I.	$\mu=1$	${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;1)$ $={}_{H^{(m)}}Bl_n(x,y;\lambda)$	Gould-Hopper Boole polynomials (GHBlP)	$\Big(1+(1+t)^\lambda\Big)^{-1}e^{x\ln(1+t)+y(\ln(1+t))^m}=\sum\limits_{n=0}^{\infty}{}_{H^{(m)}}Bl_n(x,y;\lambda)\frac{t^n}{n!}$, ${}_{H^{(m)}}Bl_n(x,y;\lambda)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}Bl_{n-q}(\lambda)S_1(q,k)\,H_k^{(m)}(x,y)$
II.	$\lambda=1$, $\mu=1$	${}_{H^{(m)}}\mathcal{S}_n(x,y;1;1)$ $={}_{H^{(m)}}Ch_n(x,y)$	Gould-Hopper Changee polynomials (GHChP)	$\Big(2+t)\Big)^{-1}e^{x\ln(1+t)+y(\ln(1+t))^m}=\sum\limits_{n=0}^{\infty}{}_{H^{(m)}}Ch_n(x,y)\frac{t^n}{n!}$, ${}_{H^{(m)}}Ch_n(x,y)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}Ch_{n-q}S_1(q,k)H_k^{(m)}(x,y)$

these polynomials and present their generating functions and series definitions in Table 4.

We note that for $m=2$ the results derived above for the GHBlP ${}_{H^{(m)}}Bl_n(x,y;\lambda)$ and GHChP ${}_{H^{(m)}}Ch_n(x,y)$ give the corresponding results for the 2-variable Hermite Boole polynomials (2VHBlP) ${}_HBl_n(x,y;\lambda)$ and 2-variable Hermite Changee polynomials (2VHChP) ${}_HCh_n(x,y)$, respectively. Again, for $m=2$, $x\to 2x$ and $y=-1$, we get the corresponding results for the Hermite Boole polynomials (HBlP) ${}_HBl_n(x;\lambda)$ and Hermite Changee polynomials (HChP) ${}_HCh_n(x)$. Also in the results of the mixed special polynomials with $H_n^{(m)}(x,y)$ as base, we obtain the results (with corresponding changes in values of indices and variable) for the corresponding mixed special polynomials with $H_n(x,y)$ and $H_n(x)$ as base.

Example 3.2. Taking $\phi(y,t)=C_0(-yt^m)$ (for which the 2VGP $p_n(x,y)$ reduce to the 2VGLP ${}_mL_n(y,x)$ Table 1(III)) in the l.h.s. of generating function (2.1), we find that the resultant 2-variable generalized Laguerre Peters polynomials (2VGLPP), denoted by ${}_{mL}\mathcal{S}_n(y,x;\lambda;\mu)$ in the r.h.s. are defined by the following generating function:

$$\Big(1+(1+t)^\lambda\Big)^{-\mu}e^{x\ln(1+t)}C_0(-y(\ln(1+t))^m)=\sum_{n=0}^{\infty}{}_{mL}\mathcal{S}_n(y,x;\lambda;\mu)\frac{t^n}{n!}. \tag{3.4}$$

The series definitions and other results for the 2VGLPP ${}_{mL}\mathcal{S}_n(y,x;\lambda;\mu)$ are given in Table 5.

Remark 3.3. Since for $m=1$ and $y\to -y$, the 2VGLP ${}_mL_n(y,x)$ reduce to the 2VLP $L_n(y,x)$ (Table 1(IV)). Therefore, taking $m=1$ and $y\to -y$ in Eq. (3.4), we get the following generating function for the 2-variable Laguerre Peters polynomials (2VLPP), denoted by ${}_L\mathcal{S}_n(y,x;\lambda;\mu)$

$$\Big(1+(1+t)^\lambda\Big)^{-\mu}e^{x\ln(1+t)}C_0(y\ln(1+t))=\sum_{n=0}^{\infty}{}_L\mathcal{S}_n(y,x;\lambda;\mu)\frac{t^n}{n!}. \tag{3.5}$$

The series definitions and other results for the 2VLPP ${}_L\mathcal{S}_n(y,x;\lambda;\mu)$ can be obtained by taking $m=1$ and $y\to -y$ in the results given in Table 5.

Table 5: Results for ${}_{m L}\mathcal{S}_n(y,x;\lambda;\mu)$.

S.No.	Results	Expressions
I.	**Series definition**	${}_{m L}\mathcal{S}_n(y,x;\lambda;\mu)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}\mathcal{S}_{n-q}(\lambda;\mu)S_1(q,k)_m L_k(y,x)$
II.	**Multiplicative and derivative operators**	$\hat{M}_{{}_m L\mathcal{S}}=\left(x+m\partial_y^{-1}\partial_x^{m-1}-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\right)\frac{1}{e^{\partial_x}}$ $\hat{P}_{{}_m L\mathcal{S}}=e^{\partial_x}-1$
III.	**Differential equation**	$\left(\frac{x(e^{\partial_x}-1)}{e^{\partial_x}}+\frac{m\partial_y^{-1}\partial_x^{m-1}(e^{\partial_x}-1)}{e^{\partial_x}}-\frac{\lambda\mu e^{(\lambda-1)\partial_x}(e^{\partial_x}-1)}{1+e^{\lambda\partial_x}}\right)-n\Big)_{m L}\mathcal{S}_n(x,y;\lambda;\mu)=0$
IV.	**Operational rule**	${}_{m L}\mathcal{S}_n(y,x;\lambda;\mu)=\mathcal{S}_n(\hat{M}_{{}_m L};\lambda;\mu)$; $\hat{M}_{{}_m L}$:=Multiplicative operator of ${}_m L_n(y,x)$
V.	**Summation formulae**	${}_{m L}\mathcal{S}_n(y+w,x;\lambda;\mu)=\sum\limits_{k=0}^{n}\binom{n}{k}{}_{m L}\mathcal{S}_k(y,x;\lambda;\mu)w^{n-k}$ ${}_{m L}\mathcal{S}_{n+k}(z,x;\lambda;\mu)=\sum\limits_{l,m=0}^{n,k}\frac{n!\,k!\,(z-y)_{l+m\,m L}\mathcal{S}_{n+k-l-m}(y,x;\lambda;\mu)}{(n-l)!\,(k-m)!\,l!\,m!}$ ${}_{m L}\mathcal{S}_n(x+w;\lambda;\mu)=\sum\limits_{k=0}^{\infty}\sum\limits_{q=k}^{n}\binom{n}{q}\mathcal{S}_k(x;\lambda;\mu)S_1(q,k)\;{}_m L_{n-q}(y,w)$

Remark 3.4. Since for $x=1$, the 2VLP $L_n(y,x)$ reduce to the classical Laguerre polynomials $L_n(y)$ [1]. Therefore, taking $x=1$ in Eq. (3.5), we get the following generating function for the Laguerre Peters polynomials (LPP), denoted by ${}_L\mathcal{S}_n(y;\lambda;\mu)$

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}e^{\ln(1+t)}C_0(y\ln(1+t))=\sum_{n=0}^{\infty}{}_L\mathcal{S}_n(y;\lambda;\mu,)\frac{t^n}{n!}. \tag{3.6}$$

The series definitions and other results for the LPP ${}_L\mathcal{S}_n(y;\lambda;\mu)$ can be obtained by taking $m=1$, $y\to -y$ and $x=1$ in the results given in Table 5.

Taking suitable values of the parameters in the results of the 2VGLPP ${}_{m L}\mathcal{S}_n(y,x;\lambda;\mu)$ and in view of Eqs. (1.15a) and (1.15b), we can find the corresponding results for the mixed special polynomials related to ${}_{m L}\mathcal{S}_n(y,x;\lambda;\mu)$. We use the suitable notations for these polynomials and present their generating functions and series definitions in Table 6.

We note that for $m=1$, $y\to -y$ the results derived above for the 2VGLBlP ${}_{m L}Bl_n(y,x;\lambda)$ and 2VGLChP ${}_{m L}Ch_n(y,x)$ give the corresponding results for the 2-variable Laguerre Boole polynomials (2VLBlP) ${}_L Bl_n(x,y;\lambda)$ and 2-variable Laguerre Changee polynomials (2VLChP) ${}_L Ch_n(x,y)$, respectively. Again, for $m=1$, $y\to -y$ and $x=1$, we get the corresponding results for the Laguerre Boole polynomials (LBlP) ${}_L Bl_n(x;\lambda)$ and Laguerre Changee polynomials (LChP) ${}_L Ch_n(x)$. Thus, in the results of the mixed special polynomials with ${}_m L_n(y,x)$ as base, we obtain the results (with corresponding changes in values of parameters) for the corresponding mixed special polynomials with $L_n(y,x)$ and $L_n(y)$ as base.

Example 3.3. By taking $\phi(y,t)=\frac{1}{1-yt^r}$ (for which the 2VGP $p_n(x,y)$ reduce to the 2VTEP $e_n^{(r)}(x,y)$ Table 1(V)) in the l.h.s. of generating function (2.1), we find that the resultant 2-variable truncated exponential Peters polynomials (2VTEPP), denoted by ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$ in the r.h.s. are defined by the following generating function:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}\Big(\frac{e^{x\ln(1+t)}}{1-y(\ln(1+t))^r}\Big)=\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{3.7}$$

The series definitions and other results for the 2VTEPP ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$ are given in Table 7.

Remark 3.5. Since for $r=2$, the 2VTEP $e^{(r)}(x,y)$ of order r reduce to the 2VTEP ${}_{[2]}e_n(x,y)$ (Table 1(VI)). Therefore, taking $r=2$ in Eq. (3.7), we get the following generating func-

Table 6: Special cases of the 2VGLPP ${}_{mL}\mathcal{S}_n(y,x;\lambda;\mu)$.

S. No.	Values of the parameters	Relation between the 2VGLPP ${}_{mL}\mathcal{S}_n(y,x;\lambda;\mu)$ and its special case	Name of the resultant special polynomials	Generating function and series definition of the resultant special polynomials
I.	$\mu=1$	${}_{mL}\mathcal{S}_n(y,x;\lambda;1)$ $={}_{mL}Bl_n(y,x;\lambda)$	2-variable generalized Laguerre Boole polynomials (2VGLBlP)	$\Big(1+(1+t))^{\lambda}\Big)^{-1}e^{x\ln(1+t)}C_0(-y(\ln(1+t))^m)=\sum_{n=0}^{\infty}{}_{mL}Bl_n(y,x;\lambda)\frac{t^n}{n!}$, ${}_{mL}Bl_n(y,x;\lambda)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Bl_{n-q}(\lambda)S_1(q,k)_mL_k(y,x)$
II.	$\lambda=1$, $\mu=1$,	${}_{mL}\mathcal{S}_n(y,x;1;1)$ $={}_{mL}Ch_n(y,x)$	2-variable generalized Laguerre Changee polynomials (2VGLChP)	$\Big(2+t)\Big)^{-1}e^{x\ln(1+t)}C_0(-y(\ln(1+t))^m)=\sum_{n=0}^{\infty}{}_{mL}Ch_n(y,x)\frac{t^n}{n!}$, ${}_{mL}Ch_n(y,x)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Ch_{n-q}S_1(q,k)_mL_k(y,x)$

Table 7: Results for ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$.

S.No.	Results	Expressions
I.	**Series definition**	${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}\mathcal{S}_{n-q}(\lambda;\mu)S_1(q,k)\,e_k^{(r)}(x,y)$
II.	**Multiplicative and derivative operators**	$\hat{M}_{e^{(r)}\mathcal{S}}=\Big(x+ry\partial_y y\partial_x^r-\frac{\lambda\mu e^{\lambda\partial_x}}{1+e^{\lambda\partial_x}}\Big)\frac{1}{e^{\partial_x}}$ $\hat{P}_{e^{(r)}\mathcal{S}}=e^{\partial_x}-1$
III.	**Differential equation**	$\Big(\frac{x(e^{\partial_x}-1)}{e^{\partial_x}}+\frac{ry\partial_y y\partial_x^r(e^{\partial_x}-1)}{e^{\partial_x}}-\frac{\lambda\mu e^{(\lambda-1)\partial_x}(e^{\partial_x}-1)}{1+e^{\lambda\partial_x}}-n\Big){}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)=0$
IV.	**Operational rule**	${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)=\mathcal{S}_n(\hat{M}_{e^{(r)}};\lambda;\mu)$; $\hat{M}_{e^{(r)}}$:=Multiplicative operator of $e_n^{(r)}(x,y)$
V.	**Summation formulae**	${}_{e^{(r)}}\mathcal{S}_n(x+w,y;\lambda;\mu)=\sum_{k=0}^{n}\binom{n}{k}{}_{e^{(r)}}\mathcal{S}_k(x,y;\lambda;\mu)(w)_{n-k}$ ${}_{e^{(r)}}\mathcal{S}_{n+k}(z,y;\lambda;\mu)=\sum_{l,m=0}^{n,k}\frac{n!\,k!\,(z-x)_{l+m}\,{}_{e^{(r)}}\mathcal{S}_{n+k-l-m}(x,y;\lambda;\mu)}{(n-l)!\,(k-m)!\,l!\,m!}$ ${}_{e^{(r)}}\mathcal{S}_n(x+w;\lambda;\mu)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}\mathcal{S}_k(x;\lambda;\mu)S_1(q,k)\,e_{n-q}^{(r)}(w,y)$

tion for the 2-variable truncated exponential Peters polynomials (2VTEPP), denoted by ${}_{[2]e}\mathcal{S}_n(x,y;\lambda;\mu)$:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}\Big(\frac{e^{x\ln(1+t)}}{1-y(\ln(1+t))^2}\Big)=\sum_{n=0}^{\infty}{}_{[2]e}\mathcal{S}_n(x,y;\lambda;\mu)\frac{t^n}{n!}. \tag{3.8}$$

The series definitions and other results for the 2VTEPP ${}_{[2]e}\mathcal{S}_n(x,y;\lambda;\mu)$ can be obtained by taking $r=2$ in the results given in Table 7.

Remark 3.6. Since for y=1, the ${}_{[2]}e_n(x,y)$ reduce to the truncated exponential polynomials ${}_{[2]}e_n(x)$ [6]. Therefore, taking $y=1$ in Eq. (3.8), we get the following generating function for the truncated exponential Peters polynomials (TEPP), denoted by ${}_{[2]e}\mathcal{S}_n(x;\lambda;\mu)$:

$$\Big(1+(1+t)^{\lambda}\Big)^{-\mu}\Big(\frac{e^{x\ln(1+t)}}{1-(\ln(1+t))^2}\Big)=\sum_{n=0}^{\infty}{}_{[2]e}\mathcal{S}_n(x;\lambda;\mu)\frac{t^n}{n!}. \tag{3.9}$$

The series definitions and other results for the TEPP ${}_{[2]e}\mathcal{S}_n(x;\lambda;\mu)$ can be obtained by taking $r=2$ and $y=1$ in the results given in Table 7.

Taking suitable values of the parameters in the results of the 2VTEPP ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$ and in view of Eqs. (1.15a) and (1.15b), we can find the corresponding results for the mixed special polynomials related to ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$. We use the suitable notations for these polynomials and present their generating functions and series definitions in Table 8.

It is also important to observe that taking $r=2$ in the results of the mixed special polynomials with $e_n^{(r)}(x,y)$ as base, we obtain the results for the corresponding mixed special polynomials with ${}_{[2]}e_n(x,y)$ as base. Also, taking $r=2$ and $y=1$, we obtain the results for the mixed special polynomials with ${}_{[2]}e_n(x)$ as base.

Further, we note that the multiplicative and derivative operators, differential equations, operational rule and summation formulae for the polynomials mentioned in Table

Table 8: Special cases of the 2VTEPP ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$.

S. No.	**Values of the parameters**	**Relation between the 2VTPP** ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$ **and its special case**	**Name of the resultant Special polynomials**	**Generating function and series definition of the resultant special polynomials**
I.	$\mu=1$	${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;1)$ $={}_{e^{(r)}}Bl_n(x,y;\lambda)$	2-variable truncated exponential Boole polynomials (2VTBlP)	$\Big(1+(1+t)^{\lambda}\Big)^{-1}\Big(\frac{e^{x\ln(1+t)}}{1-y(\ln(1+t))^r}\Big)=\sum_{n=0}^{\infty}{}_{e^{(r)}}Bl_n(x,y;\lambda)\frac{t^n}{n!}$ ${}_{e^{(r)}}Bl_n(x,y;\lambda)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Bl_{n-q}(\lambda)S_1(q,k)e_k^{(r)}(x,y)$
II.	$\lambda=1$, $\mu=1$	${}_{e^{(r)}}\mathcal{S}_n(x,y;1;1)$ $={}_{e^{(r)}}Ch_n(x,y)$	2-variable truncated exponential Changee polynomials (2VTChP)	$(2+t)^{-1}\Big(\frac{e^{x\ln(1+t)}}{1-y(\ln(1+t))^r}\Big)=\sum_{n=0}^{\infty}{}_{e^{(r)}}Ch_n(x,y)\frac{t^n}{n!}$, ${}_{e^{(r)}}Ch_n(x,y)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}Ch_{n-q}S_1(q,k)e_k^{(r)}(x,y)$

??, Table 6 and Table 8 can also be obtained by taking suitable values of the parameters in the corresponding results of the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$, 2VGLPP ${}_{mL}\mathcal{S}_n(x,y;\lambda;\mu)$ and 2VTEPP ${}_{e^{(r)}}\mathcal{S}_n(x,y;\lambda;\mu)$, respectively.

In the next section, numbers related to some mixed special polynomials are explored.

4 Concluding remarks

We know that the Peters, Boole and changee numbers have deep connections with number theory and occur in combinatorics. These numbers appear as special values of the Peters, Boole and changee polynomials as indicated in Eq. (1.20).

The Peter numbers $\mathcal{S}_n(\lambda;\mu)$ are defined by the generating function (1.18a). In view of relations (1.15a) and (1.15b), we find the following special cases of $\mathcal{S}_n(\lambda;\mu)$:

$$S_n(\lambda,1)=Bl_n(\lambda), \tag{4.1a}$$

$$S_n(1,1)=Ch_n(1)=Ch_n, \tag{4.1b}$$

Here, we explore the numbers corresponding to the HPP ${}_H\mathcal{S}_n(x;\lambda;\mu)$ defined by the generating function (3.3). Taking $m=2$, $y=-1$ and replacing x by $2x$ in series definition of the GHPP ${}_{H^{(m)}}\mathcal{S}_n(x,y;\lambda;\mu)$ (Table 3(I)), we find the following series definition of the HPP ${}_H\mathcal{S}_n(x;\lambda;\mu)$

$${}_H\mathcal{S}_n(x;\lambda;\mu)=\sum_{k=0}^{\infty}\sum_{q=k}^{n}\binom{n}{q}\mathcal{S}_{n-q}(\lambda;\mu)S_1(q,k)H_k(x). \tag{4.2}$$

Since, the HPP ${}_H\mathcal{S}_n(x;\lambda;\mu)$ are defined in terms of the Hermite polynomials. Therefore, in order to find the Hermite Peters numbers, denoted by ${}_H\mathcal{S}_n(\lambda;\mu)$, we require the Hermite numbers.

We recall that the Hermite numbers H_n are the values of the Hermite polynomials $H_n(x)$ at zero argument, that is

$$H_n:=H_n(0). \tag{4.3}$$

From the generating function of the Hermite polynomials

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \tag{4.4}$$

it follows that

$$\exp(-t^2)=\sum_{n=0}^{\infty}H_n\frac{t^n}{n!}. \tag{4.5}$$

A closed formula for H_n is given as:

$$H_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{(-1)^{n/2} n!}{\left(\dfrac{n}{2}\right)!}, & \text{if } n \text{ is even.} \end{cases} \tag{4.6}$$

Now, taking $x=0$ in both sides of definition (4.2) and using the notation

$$_H\mathcal{S}_n(0;\lambda;\mu) := {}_H\mathcal{S}_n(\lambda;\mu) \tag{4.7}$$

in the l.h.s. and notation (4.3) in the r.h.s. of the resultant equation, we find that the Hermite Peters numbers $_H\mathcal{S}_n(\lambda;\mu)$ are defined as:

$$_H\mathcal{S}_n(\lambda;\mu) = \sum_{k=0}^{\infty} \sum_{q=k}^{n} \binom{n}{q} \mathcal{S}_{n-q}(\lambda;\mu) S_1(q,k) H_k. \tag{4.8}$$

Taking $\mu=1$ in Eq. (4.8) and using relation (4.1a) in the r.h.s. and denoting the resultant Hermite Boole numbers in the l.h.s. by $_H Bl_n(\lambda)$, that is

$$_H Bl_n(\lambda) := {}_H\mathcal{S}_n(\lambda;1), \tag{4.9}$$

we find the following series definition of the $_H Bl_n(\lambda)$:

$$_H Bl_n(\lambda) = \sum_{k=0}^{\infty} \sum_{q=k}^{n} \binom{n}{q} Bl_{n-q}(\lambda) S_1(q,k) H_k. \tag{4.10}$$

Next, taking $\lambda=1$ and $\mu=1$ in Eq. (4.8) and using relation (4.1b) in the r.h.s. and denoting the resultant Hermite Changee numbers, in the l.h.s. by $_H Ch_n$, that is

$$_H Ch_n := {}_H\mathcal{S}_n(1;1), \tag{4.11}$$

we find the following series definition of the $_H Ch$:

$$_H Ch_n = \sum_{k=0}^{\infty} \sum_{q=k}^{n} \binom{n}{q} Ch_{n-q} S_1(q,k) H_k. \tag{4.12}$$

In our next investigation, we derive certain results for the 2-variable Apostol type and related polynomials by means of umbral technique.

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