Analysis in Theory and Applications
Anal. Theory Appl., Vol. **34**, No. 2 (2018), pp. 175-186
DOI: 10.4208/ata.2018.v34.n2.7

Inequalities Concerning The Maximum Modulus of Polynomials

B. A. Zargar*, A. W. Manzoor and Shaista Bashir

Department of Mathematics, University of Kashmir, Srinagar-190006, India

Received 27 July 2017; Accepted (in revised version) 15 December 2017

Abstract. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number β, with $|\beta|\leq 1$ and $R\geq 1$, it was shown by A.Zireh et al. [7] that for $|z|=1$,

$$\min_{|z|=1}\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^n P(z)\right|\geq k^{-n}\left|R^n+\beta\left(\frac{R+k}{1+k}\right)^n\right|\min_{|z|=k}|P(z)|.$$

In this paper, we shall present a refinement of the above inequality. Besides, we shall also generalize some well-known results.

Key Words: Growth of polynomials, minimum modulus of polynomials, inequalities.

AMS Subject Classifications: 30A10, 30C10, 30E15

1 Introduction and statement of results

If $P(z)$ is a polynomial of degree n then concerning the estimate of $|P(z)|$ on the disk $|z|=R$, $R>0$, we have the following inequalities

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)| \tag{1.1}$$

and

$$\max_{|z|=r<1}|P(z)|\geq r^n\max_{|z|=1}|P(z)|. \tag{1.2}$$

Inequality (1.1) is a simple consequence of Maximum Modulus Principle [5] where as inequality (1.2) is due to Zarantonillo and Verga [6]. Both the inequalities are sharp and equality holds for $P(z)=\lambda z^n$, $|\lambda|=1$.

*Corresponding author. *Email addresses:* bazargar@gmail.com (B. A. Zargar), manzoorw9@gmail.com (A. W. Manzoor), pandithsha@gmail.com (S. Bashir)

For polynomials having no zero in $|z|<1$, an inequality analogus to (1.1) due to Ankeny and Rivilin [1] is the following:

$$\max_{|z|=R}|P(z)|\leq\frac{R^n+1}{2}\max_{|z|=1}|P(z)|,\quad R\geq 1. \tag{1.3}$$

The inequality is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$.

As a refinement of inequality (1.3) Aziz and Dawood [3] have found that

If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R\geq 1$,

$$\max_{|z|=R}|P(z)|\leq\left(\frac{R^n+1}{2}\right)\max_{|z|=1}|P(z)|-\left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|. \tag{1.4}$$

The result is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta$ with $|\alpha|=|\beta|$.

Recently, A.Zireh et al. [7] have generalised inequality (1.4) and some results due to Dewan and Hans [4]. In fact they have considered the zeros of largest moduli and proved the following results.

Theorem 1.1. *Let $P(z)$ be a polynomial of degree n, having all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$.*

$$\min_{|z|=1}\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^n P(z)\right|\geq k^{-n}\left|R^{-n}+\beta\left(\frac{R+k}{1+k}\right)^n\right|\min_{|z|=k}|P(z)|. \tag{1.5}$$

The result is best possible and equality holds for the polynomial $P(z)=\alpha\left(\frac{z}{k}\right)^n$.

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\leq 1$, then for every real or complex number β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$, we have*

$$\begin{aligned}
&\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^n P(z)\right|\\
\leq&\frac{1}{2}\Bigg\{\left(k^{-n}|R^n+\beta\left(\frac{R+k}{1+k}\right)^n|+|1+\beta\left(\frac{R+k}{1+k}\right)^n|\right)\max_{|z|=k}|P(z)|\\
&-\left(k^{-n}|R^n+\beta\left(\frac{R+k}{1+k}\right)^n|-|1+\beta\left(\frac{R+k}{1+k}\right)^n|\right)\min_{|z|=k}|P(z)|\Bigg\}.
\end{aligned} \tag{1.6}$$

The inequality (1.6) *is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta k^n$, with $|\alpha|=|\beta|$.*

In this paper, we consider the moduli of all the zeros of a polynomial and present some interesting results which provide refinements of Theorems A and B. we shall also generalize some well known results.

First, we shall prove the following refinement of Theorem 1.1.

Theorem 1.3. *Let $P(z)$ be a polynomial of degree n and $z_1,z_2,\cdots,z_n$ are its zeros such that $|z_j|\le k$, $k\le 1$, then for every real or complex number β with $|\beta|\le 1$, $R\ge 1$ and $|z|=1$, we have*

$$\min_{|z|=1}|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)|\ge k^{-n}|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)|\min_{|z|=k}|P(z)|. \tag{1.7}$$

The result is best possible and equality holds for $P(z)=\alpha(\frac{z}{k})^n$.

If we take $\beta=0$ in Theorem 1.3, we get the following elegant result:

Corollary 1.1. If $P(z)$ is a polynomial of degree n, having all its zeros in $|z|\le k$, $k\le 1$, then for $R\ge 1$,

$$k^n\min_{|z|=R}|P(z)|\ge R^n\min_{|z|=k}|P(z)|.$$

The result is best possible and equality holds for $P(z)=\alpha(\frac{z}{k})^n$.

Remark 1.1. To show Theorem 1.3 is a refinement of Theorem 1.1 we shall transfer the terms with $\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}$ on the same side of inequality (1.7) and get

$$\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\left(\min_{|z|=1}|P(z)|+k^{-n}\min_{|z|=k}|P(z)|\right)\ge\frac{1}{k^n}\left(R^n\min_{|z|=k}|P(z)|-k^n\min_{|z|=R}|P(z)|\right),$$

since

$$\frac{R+|z_j|}{1+|z_j|}\ge\frac{R+k}{1+k},$$

that is, $k(R-1)\ge|z_j|(R-1)$ or, $|z_j|\le k$, which is trivially true and hence Theorem 1.3 is a refinement of Theorem 1.1.

Next,we shall present the following refinement of Theorem 1.2.

Theorem 1.4. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\le 1$, then for every real or complex number β with $|\beta|\le 1$, $R\ge 1$ and for $|z|=1$, we have,*

$$\begin{aligned}&|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)|\\ \le&\frac{1}{2}\Bigg\{\left(k^{-n}|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)|+|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)|\right)\max_{|z|=k}|P(z)|\\ &-\left(k^{-n}|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)|-|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)|\right)\min_{|z|=k}|P(z)|\Bigg\}.\end{aligned} \tag{1.8}$$

The inequality (1.8) *is sharp and equality holds for $P(z)=\alpha z^n+\beta k^n$, with $|\alpha|=|\beta|$.*

If we take $\beta=0$ in Theorem 1.4, we get the following generalization of inequality (1.4).

Corollary 1.2. If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\leq 1$ then for $R\geq 1$,

$$\max_{|z|=1}|P(z)|\leq\left(\frac{R^n+k^n}{2k^n}\right)\max_{|z|=k}|P(z)|-\left(\frac{R^n-k^n}{2k^n}\right)\min_{|z|=k}|P(z)|. \tag{1.9}$$

The inequality (1.9) is sharp and equality holds for $P(z)=\alpha z^n+\beta k^n$, $|\alpha|=|\beta|$.

2 Lemmas

For the proof of these Theorems we need the following Lemmas. Lemma 2.1 is of independent interest.

Lemma 2.1. *If $P(z)$ is a polynomial of degree n, such that $P(z)$ does not vanish in $|z|<k$, $k\geq 1$, if $z_1,z_2,\cdots,z_n$ are the zeros of $P(z)$ then for $r\leq 1$.*

$$|P(z)|\geq\prod_{j=1}^{n}\frac{(r+|z_j|)}{1+|z_j|}|P(z)|, \quad for \quad |z|=1, \tag{2.1}$$

The result is best possible and equality holds for $P(z)=(z+k)^n$.

Proof of Lemma 2.1. Since all the zeros of $P(z)$ lie in $|z|\geq k, k\geq 1$, we write

$$P(z)=c\prod_{j=0}^{n}(z-R_je^{i\theta_j}),$$

where $R_j\geq k$, $j=1,2,\cdots,n$. Therefore, for $0\leq\theta<2\pi$, $r\leq 1$, we have

$$\begin{aligned}\left|\frac{P(re^{i\theta})}{P(e^{i\theta})}\right|&=\prod_{j=1}^{n}\left|\frac{re^{i\theta}-R_je^{i\theta_j}}{e^{i\theta}-R_je_j^{i\theta}}\right|=\prod_{j=1}^{n}\left|\frac{re^{i(\theta-\theta_j)}-R_j}{e^{i(\theta-\theta_j)}-R_j}\right|\\&=\prod_{j=1}^{n}\left\{\frac{r^2+R_j^2-2rR_jcos(\theta-\theta_j)}{1+R_j^2-2R_jcos(\theta-\theta_j)}\right\}^{\frac{1}{2}}\\&\geq\prod_{j=1}^{n}\frac{r+R_j}{1+R_j}=\prod_{j=1}^{n}\frac{r+|z_j|}{1+|z_j|}.\end{aligned}$$

Which implies

$$|P(rz)|\geq\prod_{j=1}^{n}\frac{r+|z_j|}{1+|z_j|}|P(z)|,$$

or,

$$\max_{|z|=r<1}|P(z)|\geq\prod_{j=1}^{n}\frac{r+|z_j|}{1+|z_j|}\max_{|z|=1}|P(z)|.$$

Thus, we complete the proof. □

Applying Lemma 2.1 to the polynomial $z^n\overline{P(\frac{1}{\bar{z}})}$, we get

Lemma 2.2. *If $P(z)$ is a polynomial of degree n having all its zeros in the disk $|z|<k$, $k\leq 1$ then for every $R\geq 1$.*

$$|P(Rz)|\geq\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)|P(z)|,\quad \text{for}\quad |z|=1. \tag{2.2}$$

the result is best possible and equality holds for $P(z)=(z+k)^n$. This result is a refinement of a result due to Aziz [2].

Lemma 2.3. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$ and $P(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|P(z)|\leq|F(z)|$ for $|z|=k$, $k\leq 1$, and if $z_1,z_2,\cdots,z_n$ are the zeros of $P(z)$ then for every real or complex number β with $|\beta|\leq 1$ and $|z|=1$, $R\geq 1$,*

$$\left|P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z)\right|\leq\left|F(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z)\right|. \tag{2.3}$$

Proof of Lemma 2.2. Let $F(z)$ be a polynomial of degree n having all its zeros in $|z|<k$, $k\leq 1$ and $P(z)$ be a polynomial of degree not exceeding that of $F(z)$ therefore it follows from Rouches theorem that for every α with $|\alpha|<1$, $F(z)+\alpha P(z)$ has as many zeros in $|z|<k$ as $F(z)$ and so has all its zeros in $|z|<k$. Since $|P(z)|\leq|F(z)|$ for $|z|=k$, therefore any zero of $F(z)$ that lies in $|z|=k$, is the zero of $P(z)$. Thus $F(z)+\alpha P(z)$ has all its zeros in $|z|\leq k$., By Lemma 2.2, it yields that for every α with $|\alpha|<1$ and $|z|=1$, $R\geq 1$,

$$\left|F(Rz)+\alpha P(Rz)\right|\geq\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}\left|F(z)+\alpha P(z)\right|,$$

therefore for any β with $|\beta|<1$, we have

$$F(Rz)+\alpha P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}(F(z)+\alpha P(z))\neq 0,$$

that is for $|z|=1$,

$$A(z)=F(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z)+\alpha\left(P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z)\right)\neq 0. \tag{2.4}$$

Hence for an appropriate choice of the argument α, we get

$$\left|F(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z)\right|\neq|\alpha|\left|P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z)\right|.$$

Thus it follows from inequality (2.4) fo $|z|=1$ that

$$\left|F(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z)\right|\geq\left|P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z)\right|. \tag{2.5}$$

If the inequality (2.5) is not true, then there is a point $z=z_0$ with $|z_0|=1$ such that for $R\geq 1$,

$$\left|F(Rz_0)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z_0)\right|<\left|P(Rz_0)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z_0)\right|,$$

we take

$$\alpha=\frac{F(Rz_0)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}F(z_0)}{P(Rz_0)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z_0)},$$

then $|\alpha|<1$ and with this choice of α, we have from (2.4), $A(z_0)=0$ for $|z_0|=1$. But this contradicts the fact that $A(z)\neq 0$ for $|z|=1$. Hence for every real or complex β with $|\beta|=1$, (2.5) follows. This completes the proof of Lemma 2.3. □

If we take

$$F(z)=\left(\frac{z}{k}\right)^n\max_{|z|=k}|P(z)|$$

in Lemma 2.3 we get, the following:

Lemma 2.4. *Let $P(z)$ be a polynomial of degree n, and if $z_1,z_2,\cdots,z_n$ are the zeros of $P(z)$ then for $|\beta|\leq 1$, $|z|=1$ and $R\geq 1$, we have*

$$\left|P(Rz)+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}P(z)\right|\leq k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\frac{R+|z_j|}{1+|z_j|}\right|\max_{|z|=k}|P(z)|. \tag{2.6}$$

Lemma 2.5. *Let $P(z)$ be a polynomial of degree n, and if $z_1,z_2,\cdots,z_n$ are its zeros, then for every β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$, we have*

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|+\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\\ \leq&\left\{k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|+\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right\}\max_{|z|=k}|P(z)|,\end{aligned} \tag{2.7}$$

where

$$Q(z)=\Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\overline{z}}\Big)}, \quad and \quad k\leq 1.$$

Proof. Let $M=\max_{|z|=k}|P(z)|$. By Rouche's Theorem it follows that for $|\alpha|>1$, the polynomial $G(z)=P(z)-\alpha M$ has no zeros in $|z|<k$, and hence the polynomial $H(z)=\big(\frac{z}{k}\big)^n\overline{G\big(\frac{k^2}{\overline{z}}\big)}$ has all its zeros in $|z|\leq k$ and $|G(z)|=|H(z)|$ for $|z|=k$. Applying Lemma 2.3, this implies for $|\beta|\leq 1$ and $|z|=1$, $R\geq 1$,

$$\Big|G(Rz)+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)G(z)\Big|\leq\Big|H(Rz)+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)H(z)\Big|. \tag{2.8}$$

Also

$$\begin{aligned}H(z)=&\Big(\frac{z}{k}\Big)^n\overline{G\Big(\frac{k^2}{\overline{z}}\Big)}=\Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\overline{z}}\Big)}-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM\\=&Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM.\end{aligned}$$

Therefore inequality (2.8) implies,

$$\begin{aligned}&\Big|\Big\{P(Rz)-\alpha M\Big\}+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big(P(z)-\alpha M\Big)\Big|\\\leq&\Big|\Big\{Q(Rz)-\overline{\alpha}R^n\Big(\frac{z}{k}\Big)^nM\Big\}+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\{Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^nM\Big\}\Big|,\end{aligned}$$

which gives,

$$\begin{aligned}&\Big|P(Rz)+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)P(z)-\alpha\Big\{1+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\}M\Big|\\\leq&\Big|Q(Rz)+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)Q(z)-\overline{\alpha}\Big(\frac{z}{k}\Big)^n\Big\{R^n+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\}M\Big|.\end{aligned} \tag{2.9}$$

For $|z|=k$, $|P(z)|=|Q(z)|$ so $\max_{|z|=k}|Q(z)|=M$. Applying Lemma 2.4 to $Q(z)$, we obtain for $|z|=1$,

$$\Big|Q(Rz)+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)Q(z)\Big|<|\alpha|k^{-n}\Big|R^n+\beta\prod_{j=1}^{n}\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big|M,$$

where

$$|\beta|\leq 1 \quad \text{and} \quad |\alpha|>1.$$

Choosing the argument of α suitably we get for $|z|=1$ and $|\beta|\leq 1$,

$$\begin{aligned}&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)-\overline{\alpha}\left(\frac{z}{k}\right)^n\left\{R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right\}M\right|\\ =&|\alpha|k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|M-\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|.\end{aligned}\tag{2.10}$$

Combining (2.9) and (2.10), it follows that

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-|\alpha|\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|M\\ \leq&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)-\alpha\left\{1+\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right\}M\right|\\ \leq&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)-\overline{\alpha}\left(\frac{z}{k}\right)^n\left\{R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right\}M\right|\\ =&|\alpha|k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|M-\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|.\end{aligned}$$

This gives,

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-|\alpha|\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|M\\ \leq&|\alpha|k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|M-\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|.\end{aligned}$$

Which implies,

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|+\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\\ \leq&|\alpha|\left\{k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|+\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right\}M.\end{aligned}$$

Letting $|\alpha|\to 1$, the result follows. Which completes the proof of Lemma 2.5. □

If we take $\beta=0$ in Lemma 2.5, we get the following generalization of inequality (1.3).

Corollary 2.1. Let $P(z)$ be a polynomial of degree n, then for every $R\geq 1$ and $|z|=1$, we have

$$\Big|P(Rz)\Big|+\Big|Q(Rz)\Big|\leq \frac{R^n+k^n}{k^n}\max_{|z|=k}|P(z)|,$$

where

$$Q(z)=\Big(\frac{z}{k}\Big)^n\overline{P\Big(\frac{k^2}{\overline{z}}\Big)} \quad \text{and} \quad k\leq 1.$$

Taking $\beta=0$ in Lemma 2.4, we get the following generalization of inequality (1.1).

Corollary 2.2. Let $P(z)$ be a polynomial of degree n, then for any $R\geq 1$ and $k\leq 1$,

$$K^n\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=k}|P(z)|.$$

3 Proofs of Theorems

Proof of Theorem 1.3. If $P(z)$ has a zero on $|z|=k$, then inequality (1.7) is trivial. So we assume that $P(z)$ has all its zeros in $|z|<k$. Therefore $m=\min_{|z|=k}|P(z)|>0$. For $|z|=k$ and $|\alpha|<1$, $\Big|\alpha m\big(\frac{z}{k}\big)^n\Big|<m\leq|P(z)|$. It follows by Rouche's Theorem that the polynomial $G(z)=P(z)-\alpha m$ has all its zeros in $|z|<k$. Using Lemma 2.2, it yields for $R\geq 1$, $|\alpha|<1$ and $|z|=1$,

$$\Big|P(Rz)-\alpha mR^n\Big(\frac{z}{k}\Big)^n\Big|\geq\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big|P(z)-\alpha m\Big(\frac{z}{k}\Big)^n\Big|.$$

Therefore, for $|\beta|<1$, the polynomial

$$P(Rz)-\alpha mR^n\Big(\frac{z}{k}\Big)^n+\beta\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\{P(z)-\alpha m\Big(\frac{z}{k}\Big)^n\Big\},$$

that is,

$$A(z)=\Big\{P(Rz)+\beta\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)P(z)\Big\}-\alpha m\Big(\frac{z}{k}\Big)^n\Big\{R^n+\beta\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\},$$

will have no zeros on $|z|=1$. As $|\alpha|<1$, therefore for $|\beta|<1$

$$\Big|P(Rz)+\beta\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)P(z)\Big|\geq\Big|m\Big(\frac{z}{k}\Big)^n\Big\{R^n+\beta\prod_{j=1}^n\Big(\frac{R+|z_j|}{1+|z_j|}\Big)\Big\}\Big|,$$

that is

$$\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|\geq mk^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|,\quad \text{for } |z|=1. \tag{3.1}$$

For $|\beta|=1$ the inequality follows. Hence the proof of Theorem 1.3 is completed. □

Proof of Theorem 1.4. Let $m=\min\limits_{|z|=k}|P(z)|$ therefore for every α with $|\alpha|<1$, $|P(z)|>|\alpha m|$. So by Rouche's Theorem the polynomial $G(z)=P(z)-\alpha m$ has no zero in $|z|\leq k$.

Equivalently, the polynomial

$$H(z)=\left(\frac{z}{k}\right)^n\overline{G\left(\frac{k^2}{\overline{z}}\right)}$$

has all its zeros in $|z|\leq k$ and $|G(z)|=|H(z)|$ for $|z|=k$. Applying Lemma 2.3, it follows that for $|\beta|\leq 1$ and $|z|=1$, $R\geq 1$,

$$\left|G(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)G(z)\right|\leq\left|H(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)H(z)\right|,$$

or, it gives by

$$H(z)=\left(\frac{z}{k}\right)^n\overline{G\left(\frac{k^2}{\overline{z}}\right)}=\left(\frac{z}{k}\right)^n\overline{P\left(\frac{k^2}{\overline{z}}\right)}-\overline{\alpha}m\left(\frac{z}{k}\right)^n=Q(z)-\overline{\alpha}m\left(\frac{z}{k}\right)^n,$$

that is

$$\begin{aligned}&\left|(P(Rz)-\alpha m)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)(P(z)-\alpha m)\right|\\ \leq&\left|\left\{Q(Rz)-\overline{\alpha}m\left(\frac{z}{k}\right)^n\right\}+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\left\{Q(z)-\overline{\alpha}m\left(\frac{z}{k}\right)^n\right\}\right|,\end{aligned}$$

which implies

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-|\alpha|m\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\\ \leq&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)-\overline{\alpha}m\left(\frac{z}{k}\right)^n\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right|.\end{aligned} \tag{3.2}$$

Since for $|z|=1$, $|P(z)|=|Q(z)|$, which gives

$$m=\min_{|z|=k}|P(z)|=\min_{|z|=k}|Q(z)|.$$

Applying Theorem 1.3 to the polynomial $Q(z)$, we have

$$\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\geq mk^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|, \tag{3.3}$$

where $|z|=1$ and $|\beta|\leq 1$.

Choosing argument of α suitably in (3.3), we get for $|z|=1$ and $|\beta|\leq 1$,

$$\begin{aligned}&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|-\overline{\alpha}m\left(\frac{z}{k}\right)^n\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\\ =&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|-|\alpha|mk^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|. \end{aligned}\tag{3.4}$$

Inequality (3.2) can be rewritten as

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-|\alpha|m\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\\ \leq&\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)-|\alpha|mk^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right|,\end{aligned}$$

that is

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\\ \leq&-|\alpha|\left\{k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|-\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right\}m, \quad \text{for } |z|=1.\end{aligned}$$

Letting $|\alpha|\to 1$, we get for $|z|=1$ and $R\geq 1$

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|-\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\\ \leq&-\left\{k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|-\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right\}m.\end{aligned}\tag{3.5}$$

Also by Lemma 2.5, we have for $|z|=1$ and $R\geq 1$

$$\begin{aligned}&\left|P(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)P(z)\right|+\left|Q(Rz)+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)Q(z)\right|\\ \leq&\left\{k^{-n}\left|R^n+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|+\left|1+\beta\prod_{j=1}^{n}\left(\frac{R+|z_j|}{1+|z_j|}\right)\right|\right\}\max_{|z|=k}|P(z)|.\end{aligned}\tag{3.6}$$

Combining inequalities (3.5) and (3.6), we get the required result. Hence the result follows. □

References

[1] N. C. Ankeny and T. J. Rivilin, On a Theorem of S. Bernstein, Pacific, J. Math., 5 (1955), 849–852.

[2] A. Aziz, Growth of polynomials whose zeros are within or outside a circle, Bull. Austral. Math. Soc., 35 (1987), 247–256.

[3] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, J. Approx. Theory, 54 (1988), 306–313.

[4] K. K. Dewan and S. Hans, Some Polynomial Inequalities in the complex domain, Analy. Theory Appl., 26 (2010), 1–6.

[5] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, New York, 2002.

[6] R. S. Verga, A Comparison of the successive over relaxation method and semi iteration methods using Chebyshieve Polynomials, J. Soc. Indus. Appl. Math., 5 (1957), 44.

[7] A. Zireh, E. Khojastenejhad and S. R. Musawi, Some results concerning growth of polynomials, Anal. Theory Appl., 29(1) (2013), 37–46.