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Inequalities Concerning The Maximum Modulus of Polynomials

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Abstract. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number β, with $|\beta|\leq 1$ and $R\geq 1$, it was shown by A.Zireh et al. [7] that for $|z|=1$,

$$\min_{|z|=1}\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^n P(z)\right|\geq k^{-n}\left|R^n+\beta\left(\frac{R+k}{1+k}\right)^n\right|\min_{|z|=k}|P(z)|.$$

In this paper, we shall present a refinement of the above inequality. Besides, we shall also generalize some well-known results.

Key Words: Growth of polynomials, minimum modulus of polynomials, inequalities.

AMS Subject Classifications: 30A10, 30C10, 30E15

1 Introduction and statement of results

If $P(z)$ is a polynomial of degree n then concerning the estimate of $|P(z)|$ on the disk $|z|=R$, $R>0$, we have the following inequalities

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)| \tag{1.1}$$

and

$$\max_{|z|=r<1}|P(z)|\geq r^n\max_{|z|=1}|P(z)|. \tag{1.2}$$

Inequality (1.1) is a simple consequence of Maximum Modulus Principle [5] where as inequality (1.2) is due to Zarantonillo and Verga [6]. Both the inequalities are sharp and equality holds for $P(z)=\lambda z^n$, $|\lambda|=1$.

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For polynomials having no zero in $|z|<1$, an inequality analogus to (1.1) due to Ankeny and Rivilin [1] is the following:

$$\max_{|z|=R}|P(z)|\leq\frac{R^n+1}{2}\max_{|z|=1}|P(z)|,\quad R\geq 1. \tag{1.3}$$

The inequality is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$.

As a refinement of inequality (1.3) Aziz and Dawood [3] have found that

If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R\geq 1$,

$$\max_{|z|=R}|P(z)|\leq\left(\frac{R^n+1}{2}\right)\max_{|z|=1}|P(z)|-\left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|. \tag{1.4}$$

The result is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta$ with $|\alpha|=|\beta|$.

Recently, A.Zireh et al. [7] have generalised inequality (1.4) and some results due to Dewan and Hans [4]. In fact they have considered the zeros of largest moduli and proved the following results.

Theorem 1.1. *Let $P(z)$ be a polynomial of degree n, having all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$.*

$$\min_{|z|=1}\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^nP(z)\right|\geq k^{-n}\left|R^{-n}+\beta\left(\frac{R+k}{1+k}\right)^n\right|\min_{|z|=k}|P(z)|. \tag{1.5}$$

The result is best possible and equality holds for the polynomial $P(z)=\alpha\left(\frac{z}{k}\right)^n$.

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\leq 1$, then for every real or complex number β with $|\beta|\leq 1$, $R\geq 1$ and $|z|=1$, we have*

$$\begin{aligned}&\left|P(Rz)+\beta\left(\frac{R+k}{1+k}\right)^nP(z)\right|\\ \leq&\frac{1}{2}\left\{\left(k^{-n}|R^n+\beta\left(\frac{R+k}{1+k}\right)^n|+|1+\beta\left(\frac{R+k}{1+k}\right)^n|\right)\max_{|z|=k}|P(z)|\right.\\ &\left.-\left(k^{-n}|R^n+\beta\left(\frac{R+k}{1+k}\right)^n|-|1+\beta\left(\frac{R+k}{1+k}\right)^n|\right)\min_{|z|=k}|P(z)|\right\}.\end{aligned} \tag{1.6}$$

The inequality (1.6) *is sharp and equality holds for the polynomial $P(z)=\alpha z^n+\beta k^n$, with $|\alpha|=|\beta|$.*

In this paper, we consider the moduli of all the zeros of a polynomial and present some interesting results which provide refinements of Theorems A and B. we shall also generalize some well known results.

First, we shall prove the following refinement of Theorem 1.1.