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Exact Meromorphic Stationary Solutions of the Cubic-Quintic Swift-Hohenberg Equation

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Abstract. In this paper, we study an ODE of the form

$$b_0u^{(4)}+b_1u''+b_2u+b_3u^3+b_4u^5=0, \quad '=\frac{d}{dz},$$

which includes, as a special case, the stationary case of the cubic-quintic Swift-Hohenberg equation. Based on Nevanlinna theory and Painlevé analysis, we first show that all its meromorphic solutions are elliptic or degenerate elliptic. Then we obtain them all explicitly by the method introduced in [7].

Key Words: Meromorphic solutions, Cubic-Quintic Swift-Hohenberg equation, Nevanlinna theory.

AMS Subject Classifications: 35Q53

1 Introduction

The real Swift-Hohenberg equation with a cubic-quintic nonlinearity

$$\partial_t u=au+bu^3-cu^5-d(q_0^2+\partial_x^2)^2u, \quad a,b,c,d,q_0\in\mathbb{R}, \tag{1.1}$$

has been extensively studied as a model equation to test the bifurcation of solutions of certain PDEs. For detailed results, see [12] and the references therein. Almost all the work concerning (1.1) is done by numerical method, few work has been undertaken on finding exact solutions of the stationary case of (1.1) in explicit form. Therefore the devotion of this paper to search for exact meromorphic solutions of (1.2) has both mathematical interests and physical significance. Here, meromorphic functions mean the functions

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meromorphic on the whole complex plane. For the stationary case, we have 0 on the l.h.s of (1.1) and it motivates the author to study a general ODE

$$b_0u^{(4)}+b_1u''+b_2u+b_3u^3+b_4u^5=0, \quad '=\frac{d}{dz}, \tag{1.2}$$

where $b_i\in\mathbb{C}$, $i=1,2,3,4,5$ and $b_0b_4\neq 0$. For $b_4=0$, which corresponds to real cubic Swift-Hohenberg (RCSH) equation, the meromorphic solutions of (1.2) have been studied in [6].

Recently, Kao and Knobloch [12] have studied the ODE (1.2) with two arbitrary constants b_2 and b_3. In our paper, we consider the ODE (1.2) with all the coefficients arbitrary. Compared with their work, the main differences are as follows. First we prove that (1.2) does not have any entire solutions and then we explicitly find *all* its meromorphic solutions with at least one pole on $\mathbb{C}$. In other words, we have found *all* the meromorphic solutions whether or not they have poles. In addition, by applying Proposition 2.1, it is shown that one can make use of the same method as we do in this paper to explicitly find all the (traveling wave) meromorphic solutions of many other ODEs and PDEs. In Section 4, we will present some new real solutions of the ODE (1.2) by choosing specific coefficients in (1.2).

Without loss of generality, we may assume $b_0=1$ and $b_4=-3/2$ by the transformation $u\mapsto ku$ with $k=\sqrt[4]{-3/2b_4}$. Multiplying (1.2) by u' and then integrating the resulting equation yield

$$4u'u'''-2(u'')^2+2b_1(u')^2+2b_2u^2+b_3u^4-u^6=c, \tag{1.3}$$

where $c\in\mathbb{C}$ is the integration constant.

The structure of this paper can now be explained. In Section 2, we will prove that all meromorphic solutions of the ODE (1.3) must belong to class W (like Weierstrass [9]), consisting of elliptic functions and their successive degeneracies, i.e., elliptic functions, rational functions of one exponential $\exp(kz)$, $k\in\mathbb{C}$ and rational functions of z. Here, W is chosen because Weierstrass proved that functions in class W are the only meromorphic functions satisfying an algebraic addition theorem [15, pp. 490]. The method involved here is a refinement [5] of Eremenko's method used in [8] as well as [9, 10], which is based on the local singularity analysis of meromorphic solutions of ODEs as well as the zero distribution and order of growth of meromorphic solutions. This is a very powerful method. For example, it has been used [9] to characterize all meromorphic traveling wave solutions of the Kuramoto-Sivashinsky (KS) equation. One key point of this method is that an upper bound on the number of poles of solutions to the ODEs being considered in the fundamental region $\mathcal{F}$ can be found. Here, the fundamental region $\mathcal{F}$ refers to $\mathbb{C}$, the period strip or the fundamental parallelogram corresponding to u rational, simply periodic or elliptic respectively. Then this allows us to construct explicitly all the meromorphic solutions of (1.3), as we shall do in Section 3. This can be done by either applying the subequation method [4, 14], or (as we will do in this paper) making use of the following result.