A Complement to the Valiron-Titchmarsh Theorem for Subharmonic Functions

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Abstract. The Valiron-Titchmarsh theorem on asymptotic behavior of entire functions with negative zeros has been recently generalized onto subharmonic functions with the Riesz measure on a half-line in $\mathbb{R}^n$, $n \geq 3$. Here we extend the Drasin complement to the Valiron-Titchmarsh theorem and show that if $u$ is a subharmonic function of this class and of order $0 < \rho < 1$, then the existence of the limit $\lim_{r \to \infty} \log u(r)/N(r)$, where $N(r)$ is the integrated counting function of the masses of $u$, implies the regular asymptotic behavior for both $u$ and its associated measure.

Key Words: Valiron-Titchmarsh theorem, Tauberian theorems for entire functions with negative zeros, Subharmonic functions in $\mathbb{R}^n$ with Riesz masses on a ray, associated Legendre functions on the cut.

AMS Subject Classifications: 31B05, 30D15, 30D35

1 Main result

The well-known Valiron-Titchmarsh Tauberian theorem [6] states that if an entire function $f(z)$ of non-integer order $\rho$ with negative zeros has regular behavior for $z = x > 0$, i.e., there exists the finite limit

$$\lim_{r \to \infty} r^{-\rho} \log f(r) = h,$$

then its zeros have the density $\lim_{t \to \infty} t^{-\rho} n(t) = \frac{\sin \pi \rho}{\pi} h$, where $n$ is the counting function of the zeros of $f$. In turn, this implies that the function $f$ is of completely regular growth in the entire complex plane. For the history of this result and the relevant references see, e.g., [5]. Drasin [1] proved a complementary result.

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If \( f \) is an entire function of order \( \lambda, 0 < \lambda < 1, \) with all zeros real and negative, then either one of the conditions
\[
\log \frac{M(r)}{n(r)} \to L > 0, \quad r \to \infty,
\]
or
\[
\log \frac{M(r)}{N(r)} \to L\lambda > 0, \quad r \to \infty,
\]
where \( N(r) = \int_0^r t^{-1} n(t) dt \), implies the asymptotic relation as \( r \to \infty, \)
\[
\log M(r) \sim r^\lambda \psi(r).
\]
Here \( \lambda \) is determined by the transcendental equation \( L = \pi / \sin(\pi \lambda) \) and \( \psi \) is a slowly varying function, that is, \( \psi(\sigma r) / \psi(r) \to 1 \) as \( r \to \infty \) for each fixed \( \sigma > 0 \). The relation \( a \sim b \) hereafter means the existence of the limit \( \lim_{r \to \infty} a(r) / b(r) = 1 \).

The author [5] has recently generalized the Valiron-Titchmarsh theorem onto subharmonic functions in \( \mathbb{R}^n, \ n \geq 3, \) In the present note we complement the results of [5] by extending the Drasin theorem onto the subharmonic functions in \( \mathbb{R}^n, \ n \geq 3. \) Introduce in \( \mathbb{R}^n \) spherical coordinates \( x = (r, \theta), r = |x|, \theta = (\theta_1, \cdots, \theta_{n-1}) \), such that \( x_1 = r \cos \theta_1, 0 \leq \theta_1 \leq \pi, \) and \( 0 \leq \theta_k \leq 2\pi \) for \( k = 2, 3, \cdots, n-1. \)

In the case under consideration, the subharmonic functions can be represented as [4, Eq. (4.5.16)]
\[
u(x) = \int_{\mathbb{R}^n} P_n(r, t, \theta_1) d\mu(y) + u_0(x), \quad (1.1)
\]
where \( \mu \) is the Riesz associated mass of \( \nu, \) \( u_0 \) is a subharmonic function of smaller growth than \( \nu, \) and the kernel \( P_n \) is the modified Weierstrass canonical kernel,
\[
P_n(r, t, \theta_1) = rt^{n-2} ((n-1)^2 \cos \theta_1 + rt[n + (n-2) \cos^2 \theta_1] + (n-1)t^2 \cos \theta_1).
\]
Without loss of generality, hereafter we assume \( u(0) = u_0 = 0. \) Let \( n(t) = \mu(B_t) \) be the counting function of the associated masses of \( \nu, \) where \( \overline{B_t} \) is the closed ball of radius \( t \) centered at the origin of \( \mathbb{R}^n, \) and \( N(r) = (n-2) \int_0^r t^{1-n} n(t) dt \) its average. Now we can state our result.

**Theorem 1.1.** Let \( \nu \) be a subharmonic function in \( \mathbb{R}^n, \ n \geq 3, \) of order \( \rho, 0 < \rho < 1, \) whose Riesz masses are distributed over the negative \( x_1 \)-axis. If the limit
\[
\lim_{r \to \infty} \frac{u(r)}{n(r)} = \Delta \quad (1.2)
\]
exists, then, as \( r \to \infty, \)
\[
u(x) \sim H(\theta) r^\rho \psi(r) \quad (1.3)
\]