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A Complement to the Valiron-Titchmarsh Theorem for Subharmonic Functions

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Abstract. The Valiron-Titchmarsh theorem on asymptotic behavior of entire functions with negative zeros has been recently generalized onto subharmonic functions with the Riesz measure on a half-line in $\mathbb{R}^n$, $n\geq 3$. Here we extend the Drasin complement to the Valiron-Titchmarsh theorem and show that if u is a subharmonic function of this class and of order $0<\rho<1$, then the existence of the limit $\lim_{r\to\infty}\log u(r)/N(r)$, where $N(r)$ is the integrated counting function of the masses of u, implies the regular asymptotic behavior for both u and its associated measure.

Key Words: Valiron-Titchmarsh theorem, Tauberian theorems for entire functions with negative zeros, Subharmonic functions in $\mathbb{R}^n$ with Riesz masses on a ray, associated Legendre functions on the cut.

AMS Subject Classifications: 31B05, 30D15, 30D35

1 Main result

The well-known Valiron-Titchmarsh Tauberian theorem [6] states that if an entire function $f(z)$ of non-integer order ρ with negative zeros has regular behavior for $z=x>0$, i.e., there exists the finite limit

$$\lim_{r\to\infty} r^{-\rho}\log f(r)=h,$$

then its zeros have the density $\lim_{t\to\infty}t^{-\rho}n(t)=\frac{\sin\pi\rho}{\pi}h$, where n is the counting function of the zeros of f. In turn, this implies that the function f is of completely regular growth in the entire complex plane. For the history of this result and the relevant references see, e.g., [5]. Drasin [1] proved a complementary result.

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If f is an entire function of order λ, $0<\lambda<1$, with all zeros real and negative, then either one of the conditions

$$\log\frac{M(r)}{n(r)}\to L>0, \quad r\to\infty,$$

or

$$\log\frac{M(r)}{N(r)}\to L\lambda>0, \quad r\to\infty,$$

where $N(r)=\int_0^r t^{-1}n(t)dt$, implies the asymptotic relation as $r\to\infty$,

$$\log M(r)\sim r^\lambda\psi(r).$$

Here λ is determined by the transcendental equation $L=\pi/\sin(\pi\lambda)$ and ψ is a slowly varying function, that is, $\psi(\sigma r)/\psi(r)\to 1$ as $r\to\infty$ for each fixed $\sigma>0$. The relation $a\sim b$ hereafter means the existence of the limit $\lim_{r\to\infty}a(r)/b(r)=1$.

The author [5] has recently generalized the Valiron-Titchmarsh theorem onto subharmonic functions in $\mathbb{R}^n$, $n\geq 3$. In the present note we complement the results of [5] by extending the Drasin theorem onto the subharmonic functions in $\mathbb{R}^n,n\geq 3$. Introduce in $\mathbb{R}^n$ spherical coordinates $x=(r,\theta)$, $r=|x|$, $\theta=(\theta_1,\cdots,\theta_{n-1})$, such that $x_1=r\cos\theta_1$, $0\leq\theta_1\leq\pi$, and $0\leq\theta_k\leq 2\pi$ for $k=2,3,\cdots,n-1$.

In the case under consideration, the subharmonic functions can be represented as [4, Eq. (4.5.16)]

$$u(x)=\int_{\mathbb{R}^n}P_n(r,t,\theta_1)d\mu(y)+u_0(x), \tag{1.1}$$

where μ is the Riesz associated mass of u, u_0 is a subharmonic function of smaller growth than u, and the kernel P_n is the modified Weierstrass canonical kernel,

$$P_n(r,t,\theta_1)=rt^{n-2}\big((n-1)r^2\cos\theta_1+rt[n+(n-2)\cos^2\theta_1]+(n-1)t^2\cos\theta_1\big).$$

Without loss of generality, hereafter we assume $u(0)=u_0=0$. Let $n(t)=\mu(\overline{B_t})$ be the counting function of the associated masses of u, where $\overline{B_t}$ is the closed ball of radius t centered at the origin of $\mathbb{R}^n$, and $N(r)=(n-2)\int_0^r t^{1-n}n(t)dt$ its average. Now we can state our result.

Theorem 1.1. *Let u be a subharmonic function in $\mathbb{R}^n$, $n\geq 3$, of order ρ, $0<\rho<1$, whose Riesz masses are distributed over the negative x_1-axis. If the limit*

$$\lim_{r\to\infty}\frac{u(r)}{n(r)}=\mathbf{\Delta} \tag{1.2}$$

exists, then, as $r\to\infty$,

$$u(x)\sim H(\theta)r^\rho\psi(r) \tag{1.3}$$