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Weighted Integral Means of Mixed Areas and Lengths Under Holomorphic Mappings

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Abstract. This note addresses monotonic growths and logarithmic convexities of the weighted ($(1-t^2)^\alpha dt^2$, $-\infty<\alpha<\infty$, $0<t<1$) integral means $\mathsf{A}_{\alpha,\beta}(f,\cdot)$ and $\mathsf{L}_{\alpha,\beta}(f,\cdot)$ of the mixed area $(\pi r^2)^{-\beta}A(f,r)$ and the mixed length $(2\pi r)^{-\beta}L(f,r)$ ($0\le\beta\le 1$ and $0<r<1$) of $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$ under a holomorphic map f from the unit disk $\mathbb{D}$ into the finite complex plane $\mathbb{C}$.

Key Words: Monotonic growth, logarithmic convexity, mean mixed area, mean mixed length, isoperimetric inequality, holomorphic map, univalent function.

AMS Subject Classifications: 32A10, 32A36, 51M25

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1 Introduction

From now on, $\mathbb{D}$ represents the unit disk in the finite complex plane $\mathbb{C}$, $H(\mathbb{D})$ denotes the space of holomorphic mappings $f:\mathbb{D}\to\mathbb{C}$, and $U(\mathbb{D})$ stands for all univalent functions in $H(\mathbb{D})$. For any real number α, positive number $r\in(0,1)$ and the standard area measure dA, let

$$dA_\alpha(z)=(1-|z|^2)^\alpha dA(z),\quad r\mathbb{D}=\{z\in\mathbb{D}:|z|<r\},\quad r\mathbb{T}=\{z\in\mathbb{D}:|z|=r\}.$$

In their recent paper [11], Xiao and Zhu have discussed the following area $0<p<\infty$-integral mean of $f\in H(\mathbb{D})$:

$$M_{p,\alpha}(f,r)=\Big[\frac{1}{A_\alpha(r\mathbb{D})}\int_{r\mathbb{D}}|f|^p dA_\alpha\Big]^{\frac{1}{p}},$$

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proving that $r\mapsto M_{p,\alpha}(f,r)$ is strictly increasing unless f is a constant, and $\log r\mapsto \log M_{p,\alpha}(f,r)$ is not always convex. This last result suggests such a conjecture that $\log r\mapsto \log M_{p,\alpha}(f,r)$ is convex or concave when $\alpha\le 0$ or $\alpha>0$. But, motivated by [11, Example 10, (ii)] we can choose $p=2$, $\alpha=1$, $f(z)=z+c$ and $c>0$ to verify that the conjecture is not true. At the same time, this negative result was also obtained in Wang-Zhu's manuscript [10]. So far it is unknown whether the conjecture is generally true for $p\neq 2$– see [9] for a recent development.

The foregoing observation has actually inspired the following investigation. Our concentration is the fundamental case $p=1$. To understand this new approach, let us take a look at $M_{1,\alpha}(\cdot,\cdot)$ from a differential geometric viewpoint. Note that

$$M_{1,\alpha}(f',r)=\frac{\int_{r\mathbb{D}}|f'|dA_\alpha}{A_\alpha(r\mathbb{D})}=\frac{\int_0^r\left[(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|\right](1-t^2)^\alpha dt^2}{\int_0^r(1-t^2)^\alpha dt^2}.$$

So, if $f\in U(\mathbb{D})$, then

$$(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|$$

is a kind of mean of the length of $\partial f(t\mathbb{D})$, and hence the square of this mean dominates a sort of mean of the area of $f(t\mathbb{D})$ in the isoperimetric sense:

$$\Phi_A(f,t)=(\pi t^2)^{-1}\int_{t\mathbb{D}}|f'(z)|^2dA(z)\le\left[(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||dz|\right]^2=\left[\Phi_L(f,t)\right]^2.$$

In accordance with the well-known Pólya-Szegö monotone principle [8, Problem 309] (or [2, Proposition 6.1]) and the area Schwarz's lemma in Burckel, Marshall, Minda, Poggi-Corradini and Ransford [2, Theorem 1.9], $\Phi_L(f,\cdot)$ and $\Phi_A(f,\cdot)$ are strictly increasing on $(0,1)$ unless $f(z)=a_1z$ with $a_1\neq 0$. Furthermore, $\log\Phi_L(f,r)$ and $\log\Phi_A(f,r)$, equivalently, $\log L(f,r)$ and $\log A(f,r)$, are convex functions of $\log r$ for $r\in(0,1)$, due to the classical Hardy's convexity and [2, Section 5]. Perhaps, it is worthwhile to mention that if $c>0$ is small enough then the universal cover of $\mathbb{D}$ onto the annulus $\{e^{-c\pi/2}<|z|<e^{c\pi/2}\}$:

$$f(z)=\exp\left[ic\log\left(\frac{1+z}{1-z}\right)\right]$$

enjoys the property that $\log r\mapsto\log A(f,r)$ is not convex; see [2, Example 5.1].

In the above and below, we have used the following convention:

$$\Phi_A(f,r)=\frac{A(f,r)}{\pi r^2}\quad\text{and}\quad\Phi_L(f,r)=\frac{L(f,r)}{2\pi r},$$

where under $r\in(0,1)$ and $f\in H(\mathbb{D})$, $A(f,r)$ and $L(f,r)$ stand respectively for the area of $f(r\mathbb{D})$ (the projection of the Riemannian image of $r\mathbb{D}$ by f) and the length of $\partial f(r\mathbb{D})$ (the boundary of the projection of the Riemannian image of $r\mathbb{D}$ by f) with respect to the standard Euclidean metric on $\mathbb{C}$. For our purpose, we choose a shortcut notation

$$d\mu_\alpha(t)=(1-t^2)^\alpha dt^2\quad\text{and}\quad\nu_\alpha(t)=\mu_\alpha([0,t]),\quad\forall t\in(0,1),$$