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A Local Property of Hausdorff Centered Measure of Self-Similar Sets

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Abstract. We analyze the local behavior of the Hausdorff centered measure for self-similar sets. If E is a self-similar set satisfying the open set condition, then

$$C^s(E\cap B(x,r))\le(2r)^s$$

for all $x\in E$ and $r>0$, where C^s denotes the s-dimensional Hausdorff centered measure. The above inequality is used to obtain the upper bound of the Hausdorff centered measure. As the applications of above inequality, We obtained the upper bound of the Hausdorff centered measure for some self-similar sets with Hausdorff dimension equal to 1, and prove that the upper bound reach the exact Hausdorff centered measure.

Key Words: Hausdorff centered measure, Hausdorff measure, self-similar sets.

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1 Introduction

Measure and dimension are basic concepts in fractals. Computing and estimating the dimension and measure of the fractal sets are active problems in recent years. The basic and important example of fractal measures are the Hausdorff measure H^s. Let $s\geq 0$, $\delta>0$, $E\subset R^n$, the s-dimensional Hausdorff measure of E is defined by

$$H^s(E)=\lim_{\delta\to 0}H^s_\delta(E)=\sup_{\delta>0}H^s_\delta(E), \tag{1.1}$$

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where $H^s_\delta(E)=\inf\{\sum_{i>0}|U_i|^s : E\subset\bigcup_{i>0}U_i,\ 0<|U_i|\le\delta\}$, with the infimum taken over all δ-covering of E by countable sets $\{U_i\}_{i>0}$.

The Hausdorff dimension of E is defined as

$$dim_H(E)=\sup\{s\ge 0: H^s(E)=\infty\}=\inf\{s\ge 0: H^s(E)=0\}.$$

Another two kinds of measures, the packing measure and the Hausdorff centered measure, have been used by comparison with the Hausdorff measure to study the irregularity of sets in R^n. The packing measure P^s where introduced by Tricot [13] and Taylor and Tricot [12]. In [10], Saint Raymond and Tricot, introduced the Hausdorff centered measure C^s.

We first recall the definition of the Hausdorff centered measure for the use of the paper.

Let $E\subset R^n$, $s\ge 0$, $\delta>0$, the s-dimensional Hausdorff centered measure of E is defined as

$$C^s(E)=\sup\{C^s_0(F): F\subset E\}, \tag{1.2}$$

where $C^s_0(F)=\lim_{\delta\to 0}C^s_\delta(F)$, and $C^s_\delta(F)=\inf\{\sum_{i>0}(2r_i)^s\}$, the infimum taken over all δ-covering by countable ball collection $\{B(x_i,r_i)\}_{i>0}$ with center $x_i\in F$. We call $C^s_0(F)$ the s-dimensional Hausdorff centered pre-measure of F. Notice that $C^s_0(\cdot)$ is not a measure since it is not monotone. Thus we define a measure $C^s(\cdot)$ by (1.2). The Hausdorff centered dimension of E is deduced by

$$dim_C(E)=\sup\{s\ge 0: C^s(E)=\infty\}=\inf\{s\ge 0: C^s(E)=0\}.$$

The Hausdorff measure and the packing measure have been extensively studied in the later years. In self-similar setting, there are some results in computation and theory. For example, Zhou [14] obtained the following inequality for self-similar sets

$$H^s(E\cap U)\le |U|^s, \tag{1.3}$$

where E is a self-similar set satisfies the open set condition, U is a set intersects with E. The similar results for the packing measure are also obtained in [5]. Moreover, There are some computable results for the Hausdorff and packing measures (see for example, [1–4]).

Much less is known about the Hausdorff centered measure, although it has some computable results recently (see [7–9, 15, 16]). As the Hausdorff centered measure differs by a constant factor from the Hausdorff measure, it may be used in the computation of the Hausdorff dimension.

In [6], Marta Liorente and Manuel Moran showed that if E is a self-simllar set satisfying the open set condition, then $C^s_0(E)=C^s(E)$, making unnecessary the step (1.2) in its definition above.