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# Fractional Variational Approach for Dissipative Mechanical Systems

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**Abstract.** More recently, a variational approach has been proposed by Lin and Wang for damping motion with a Lagrangian holding the energy term dissipated by a friction force. However, the modified Euler-Lagrange equation obtained within their formalism leads to an incorrect Newtonian equation of motion due to the nonlocality of the Lagrangian. In this communication, we generalize this approach based on the fractional actionlike variational approach and we show that under some simple restrictions connected to the fractional parameters introduced in the fractional formalism, this problem may be solved.

**Key Words**: Fractional actionlike variational approach, dissipative system.

**AMS Subject Classifications**: 26A33, 49S05, 70F40

## 1 Introduction

Fractional calculus has found many important applications in different branches of sciences and engineering [5, 22, 29, 30, 35, 36]. Some nice results were obtained as well in finance [13, 15, 37], stochastic processes [11, 12, 27] and probability theories [21]. Fractional derivative and integral operators are suitable for describing dynamical systems with dissipation [16, 17, 19, 34] and nonlocal systems [14]. Recently, several approaches have been developed to generalize the least action principle by including fractional derivatives in order to describe dissipative systems, nonlinear open systems and hereditary properties of a large number of materials, neveretheless, mathematical analysis of the resulting equations of motion require lots of refinement. This mathematical problem follows from the complicated chain and Leibnitzs rules used when nonlocal fractional derivatives are considered. In fact, there has been a longstanding effort to formulate

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least action principle for dissipative dynamical systems free from complicated mathematical tricks. There exist in literature different approaches to deal with nonconservative dynamical systems (the reader is referred to the reviews in [38] about the details of these approaches). Few years ago, we have introduced a simple approach recognized as the fractional actionlike variational approach based on the notion of Riemann-Liouville fractional integral with fractional parameter [16, 17]. This approach seems successful to describe a good number of dissipative dynamical systems and belongs to the class of fractional calculus of variations which received recently a particular attention due to the successful applications of the resulting fractional Euler-Lagrange equations in nonlinear systems (see [1–4, 6, 7, 9, 10, 18, 23, 28, 31, 32] and the excellent monograph [26] with references therein).

The main aim of this communication is to extend the fractional actionlike variational action approach for the case of dissipative systems. In fact, we are motivated by the work of Wang et al. [39] and Lin and Wang [24, 25] where a possible global variational principle for nonlocal dissipative systems (NDS) was constructed. In Wang et al. approach, an entire isolated conservative system containing a damped body and its environment, coupled to each other by friction is taken into account with a total Lagrangian $L_{\text{total}} = L_{\text{standard}} - E_d$, $L_{\text{standard}}$ is the standard Lagrangian and $E_d$ is the negative work of the friction force which is considered to be nonlocal. For damping motion with friction force $f_d$ and position $q$, Wang et al. define $E-d$ by $E_d = \int_0^t f(\tau)d\tau$ where $f = d_d(\tau)\dot{q}(\tau)$, $\dot{q}$, being the velocity of motion. In this approach, the action of the theory is given by

$$S = \int_0^T L_{\text{standard}} dt - \int_0^T \int_0^t f d\xi.$$

It was observed within this approach that the least action principle is comparable to a least dissipation principle for the case of Stokes damping or for over-damped motion. However, in this formalism the resulting equation of motion is not the correct Newtonian equation of motion. In this communication, we will generalize this variational approach fractionally and we will prove that more nice properties will be raised accordingly. More precisely, we will prove that under some simple restrictions connected to the fractional parameters introduced in the theory, the true equation of motion is obtained.

## 2 Main results

We start by introducing the following definition:

**Definition 2.1.** Consider a function $q \in C^1[0,t]$ and letting $L_{\text{total}}(\dot{q},q,\tau) : L_{\text{total}} \in C^2([0,t] \times \mathbb{R}^2;\mathbb{R})$ be the total Lagrangian of the previous NDS with $(\dot{q},q,\tau) \to L_{\text{total}}(\dot{q},q,\tau)$ assumed to be a $C^2$ function with respect to all its arguments. We define the fractional action in the NDS by:

$$S_{\alpha,\beta} = \frac{1}{\Gamma(\alpha)} \int_0^t L_{\text{standard}}(\dot{q},q,\tau)(t-\tau)^{\alpha-1} d\tau - \frac{1}{\Gamma(\beta)} \int_0^t \Big( \int_0^\tau f(\dot{q},q)(t-\xi)^{\beta-1} dx \Big) d\tau, \quad (2.1)$$