Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 3 (2014), pp. 318-343
DOI: 10.4208/ata.2014.v30.n3.8

Existence and Uniqueness of Renormalized Solution of Nonlinear Degenerated Elliptic Problems

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Received 6 March 2013; Accepted (in revised version) 26 August 2014

Available online 16 October 2014

Abstract. In this paper, We study a general class of nonlinear degenerated elliptic problems associated with the differential inclusion $\beta(u)-div(a(x,Du)+F(u))\ni f$ in Ω, where $f\in L^1(\Omega)$. A vector field $a(\cdot,\cdot)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in the functional spaces we prove the existence of renormalized solutions for general L^1-data. Under an additional strict monotonicity assumption uniqueness of the renormalized solution is established.

Key Words: Weighted Sobolev spaces, Hardy inequality, Truncations, maximal monotone graphe, degenerated elliptic operators.

AMS Subject Classifications: 35K45, 35K61, 35K65

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ with Lipschitz boundary if $N\geq 2$, p be a real number such that $1<p<\infty$ and $w=\{w_i(x), 0\leq i\leq N\}$ be a vector of weight functions on Ω, i.e., each $w_i(x)$ is a measurable a.e. positive on Ω. Let $W_0^{1,p}(\Omega,w)$ be the weighted Sobolev space associated with the vector w. Our aim is to show the existence and uniqueness of renormalized solutions to the following nonlinear elliptic equation

$$\begin{cases} \beta(u)-div(a(x,Du)+F(u))\ni f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with a right-hand side f which is assumed to belong either to $L^\infty(\Omega)$ or to $L^1(\Omega)$ for Eq. (1.1). Furthermore, $F:\mathbb{R}\rightarrow\mathbb{R}^{\mathbb{N}}$ is locally lipschitz continuous and $\beta:\mathbb{R}\rightarrow 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0\in\beta(0)$, Moreover, we assume that

$$\beta^0(l)\in L^1(\Omega) \tag{1.2}$$

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for each $l\in\mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_0(l)=\inf\{|r|/r\in\mathbb{R}\text{and}r\in\beta(l)\}$$

$a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

(A_1) There exists a positive constant λ such that

$$a(x,\xi)\cdot\xi\geq\lambda\sum_{i=1}^{N}w_i|\xi_i|^p$$

holds for all $\xi\in\mathbb{R}^N$ and almost every $x\in\Omega$.

(A_2) $|a_i(x,\xi)|\leq\alpha w_i^{\frac{1}{p}}(x)[k(x)+\sum_{j=1}^N w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}]$ for almost every $x\in\Omega$, all $i=1,\cdots,N$ and every $\xi\in\mathbb{R}^N$, where $k(x)$ is a nonnegative function in $L^{p'}(\Omega)$, $p'=p/(p-1)$, for a.e. $x\in\Omega$ and $\alpha>0$.

(A_3) $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq 0$ for almost every $x\in\Omega$ and every $\xi,\eta\in\mathbb{R}^N$.

We use in this paper the the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [9] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of Eq. (1.1) by L. Boccardo et al. [4] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [16] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [15] for the case of right hand side is general measure data. The equivalent notion of entropy solution has been introduced by Bénilan et al. in [6]. For results on the existence of renormalized solutions of elliptic problems of type (1.1) with $a(\cdot,\cdot)$ satisfying a variable growth condition we refer to [2,12,18] and [4]. One of the motivations for studying (1.1) comes from applications to electrorheological fluids (see [17] for more details) as an important class of non-Newtonian fluids.

The plan of the paper is as follows. In Section 2 we give some preliminaries and the definition of weighted Sobolev spaces. In Section 3 we make precise all the assumptions on a,β,F,f and we introduce the notions of weak and also renormalized solution for the problem (1.1). Our first main result, the existence of a renormalized solution to Eq. (1.1) for any L^∞-data f, are collected in Section 4. Our second main result, existence of a renormalized solution to (1.1) for any L^1-data f, and the results on uniqueness of renormalized solutions are collected in Section 5. Section 6 is devoted to an example for illustrating our abstract result.

2 Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ p be a real number such that $1<p<\infty$ and $\omega=\{\omega_i(x),\,0\leq i\leq N\}$ be a vector of weight functions, i.e., every component $\omega_i(x)$