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Some Characterizations of $VMO(\mathbb{R}^n)$

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Abstract. In this paper we give three characterizations of $VMO(\mathbb{R}^n)$ space, which are of John-Nirenberg type, Uchiyama-type and Miyachi-type, respectively.

Key Words: VMO space, John-Nirenberg inequality, multiplier, CMO space.

AMS Subject Classifications: 42B35, 42B20

1 Introduction

Suppose that f is a locally integrable function on $\mathbb{R}^n$ and $Q\subset\mathbb{R}^n$ is a cube with sides paralleling to coordinate axis. Denote by f_Q the mean of f on Q, that is,

$$f_Q=\frac{1}{|Q|}\int_Q f(x)dx.$$

For $a>0$, let

$$M_a(f)=\sup_{|Q|\le a}\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx.$$

A locally integral function f is said to belong to $BMO(\mathbb{R}^n)$ if there exists a constant $C>0$ such that $\sup_{a>0}M_a(f)\le C$. The minimal constant C is defined to be the $BMO(\mathbb{R}^n)$ norm of f and denoted by $\|f\|_*$.

In 1975, Sarason [7] defined the VMO function on $\mathbb{R}$ and gave its characterization. A function f in $BMO(\mathbb{R})$ is said to belong to $VMO(\mathbb{R})$, if

$$M_0(f):=\lim_{a\to 0}M_a(f)=0.$$

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Theorem 1.1 (see [7]). *Let f belong to $BMO(\mathbb{R})$, then the following conditions are equivalent:*

(i) $f\in VMO(\mathbb{R})$;

(ii) f is in the BMO-closure of $UC(\mathbb{R})\cap BMO(\mathbb{R})$;

(iii) $\lim_{|y|\to 0}\|\tau_y f-f\|_*=0$, where and in the sequel, $\tau_y f(x)=f(x-y)$;

(iv) $f=u+Hv$, where u and v belong to $BUC(\mathbb{R})$ and H denotes the Hilbert transform.

Here and in the sequel, $UC(\mathbb{R}^n)$ $(n\geq 1)$ denotes the space of complex valued, uniform continuous functions on $\mathbb{R}^n$ and $BUC(\mathbb{R}^n)=L^\infty(\mathbb{R}^n)\cap UC(\mathbb{R}^n)$.

Using the similar idea as proving Theorem 1.1, it is easy to prove the following variant of Theorem 1.1 in higher dimensions, we omit the details here.

Theorem 1.2. *Let f belong to $BMO(\mathbb{R}^n)$, then the following conditions are equivalent:*

(i) $f\in VMO(\mathbb{R}^n)$;

(ii) f is in the BMO-closure of $UC(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)$;

(iii) $\lim_{|y|\to 0}\|\tau_y f-f\|_*=0$;

(iv) $f=\phi_0+\sum_{j=1}^n R_j\phi_j$, where $\phi_j\in BUC(\mathbb{R}^n)$ $(j=0,1,\cdots,n)$ and R_j $(1\leq j\leq n)$ denote the Riesz transforms, that is:

$$R_jf(x)=c_n \quad p.v. \int_{\mathbb{R}^n}\frac{x_j-y_j}{|x-y|^{n+1}}f(y)dy, \quad where \;\; c_n=\Gamma\Big(\frac{n+1}{2}\Big)\pi^{-\frac{n+1}{2}}.$$

In the present paper, we will give the other characterizations of $VMO(\mathbb{R}^n)$. We first characterize $VMO(\mathbb{R}^n)$ by a John-Nirenberg type equation in Section 2. A characterization of $VMO(\mathbb{R}^n)$ of Uchiyama-type will be given in Section 3. As an application, we also give the characterization of $VMO(\mathbb{R}^n)$ of Miyachi-type in Section 4. In the last section, as a remark, we state that some results hold also for $CMO(\mathbb{R}^n)$, the BMO-closure of $C_0(\mathbb{R}^n)$, the space of all continuous functions on $\mathbb{R}^n$ which vanish at infinity.

2 John-Nirenberg type characterization of $VMO(\mathbb{R}^n)$

For $f\in L_{\rm loc}(\mathbb{R}^n)$, $\lambda>0$ and $a>0$, denote $J(f;\lambda,a)$ by

$$J(f;\lambda,a):=\sup_{|Q|\leq a}\frac{1}{|Q|}\int_Q \exp\Big(\frac{\lambda}{\|f\|_*}|f(x)-f_Q|\Big)dx,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with $|Q|\leq a$. In 1961, John and Nirenberg [4] proved that if $f\in BMO(\mathbb{R}^n)$, then there exist $\lambda>0$ and $C_1>0$, such that

$$\sup_{a>0}J(f;\lambda,a)\leq C_1,$$

which is called the John-Nirenberg inequality. In this section, we give a characterization of $VMO(\mathbb{R}^n)$ by $J(f;\lambda,a)$.