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Some Results on the Simultaneous Approximation

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Abstract. In this paper, we give some result on the simultaneous proximinal subset and simultaneous Chebyshev in the uniformly convex Banach space. Also we give relation between fixed point theory and simultaneous proximity.

Key Words: Simultaneous proximinal, simultaneous Chebyshev, fixed point.

AMS Subject Classifications: 41A50, 41A52, 46N10

1 Introduction

The modulus of convexity of a Banach space X is a function $\delta:[0,2]\to[0,1]$ defined by

$$\delta_X(t)=\inf\Big\{1-\|\frac{x+y}{2}\|:\ x,y\in X,\ \|x\|\le 1,\ \|y\|\le 1,\ \|x-y\|\ge t\Big\}.$$

The characteristic of convexity or the coefficient of convexity of X is the number

$$\epsilon_0(X)=\sup\{\varepsilon\in[0,2]:\ \delta_X(\varepsilon)=0\}.$$

Also let W be a subset of X and C a bounded set in X. We remember that a point $w_0\in W$ is said to be a best simultaneous approximation for C from W, if

$$d(C,W)=\sup_{c\in C}\|c-w_0\|,$$

where

$$d(C,W):=\inf_{w\in W}\sup_{c\in C}\|c-w\|.$$

If for each bounded set C there is at least one best simultaneous approximation in W, then W is called a *simultaneous proximinal subset* of X. Also, if for each bounded set C there is an unit best simultaneous approximation in W, then W is called a *simultaneous Chebychev subset* of X.

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Let W be a subspace of a normed linear space X, then for a bounded set C we put

$$\mathcal{S}_W(C)=\Big\{w_0\in W: d(C,W)=\sup_{c\in C}\|c-w_0\|\Big\}$$

the set of all best simultaneous approximations for C from W. It is clear that $\mathcal{S}_W(C)$ is a bounded and convex subset of X and if W is a simultaneous proximinal subset in X, then W is closed in X.

The simultaneous proximinal subset evolves as a generalization of the concept of best approximation. The reader can find some important results of it in [2,4,5].

2 Main results

Theorem 2.1. *Let W be a nonempty weakly compact and convex subset of a Banach space X. Then W is a simultaneous proximinal subset of X.*

Proof. Suppose C is a arbitrary bounded subset of X. Because W is a nonempty weakly compact and convex subset and the map $x\mapsto \sup_{s\in C}\|s-x\|$ is continuous, it follows that $\mathcal{S}_W(C)$ is nonempty and therefore W is a simultaneous proximinal subset of X. □

Theorem 2.2. *Let W be a nonempty closed convex subset of a uniformly convex Banach space X. Then W is a simultaneous Chebeyshev subset of X.*

Proof. Suppose C is an arbitrary bounded subset of X. Because the map $x\mapsto\sup_{s\in C}\|s-x\|$ is a continuous and convex functional and $\sup_{s\in C}\|s-x\|\to\infty$ as $\|x\|\to\infty$, by [1, Theorem 2.5.8], $\mathcal{S}_W(C)\neq\emptyset$. Suppose $\mathcal{S}_W(C)$ is not a singleton. We claim that

$$diam(\mathcal{S}_W(C))\le\epsilon_0(X)d(C,W).$$

Set $d=diam(\mathcal{S}_W(C))$ then $d>0$. Let $r\in(0,d)$ and $x,y\in\mathcal{S}_W(C)$ with $\|x-y\|\ge d-r$. By the convexity of $\mathcal{S}_W(C)$, $(x+y)/2\in\mathcal{S}_W(C)$. Also By the definition of modulus of convexity for $s\in C$ we have

$$\Big\|\frac{s-x}{2}+\frac{s-y}{2}\Big\|\le d(C,W)\Big[1-\delta_X\Big(\frac{\|x-y\|}{d(C,W)}\Big)\Big].$$

Therefore

$$d(C,W)=\sup_{s\in C}\Big\|s-\Big(\frac{x+y}{2}\Big)\Big\|\le d(C,W)\Big[1-\delta_X\Big(\frac{\|x-y\|}{d(C,W)}\Big)\Big].$$

Hence

$$\delta_X\Big(\frac{\|x-y\|}{d(C,W)}\Big)\le 0,$$

and so

$$\delta_X\Big(\frac{\|x-y\|}{d(C,W)}\Big)=0.$$