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Convergence of the q Analogue of Szász-Beta-Stancu Operators

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Abstract. In the present paper, we propose the q analogue of Szász-Beta-Stancu operators. By estimate the moments, we establish direct results in terms of the modulus of smoothness. Investigate the rate of point-wise convergence and weighted approximation properties of the q operators. Voronovskaja type theorem is also obtained. Our results generalize and supplement some convergence results of the q-Szász-Beta operators, thus they improve the existing results.

Key Words: q-Szász-Beta operators, q-analogues, modulus of smoothness, stancu, weighted approximation.

AMS Subject Classifications: 41A25, 41A35, 41A36

1 Introduction

For $f\in C_\gamma[0,\infty)$, a new type of Szász-Beta operator studied by Gupta and Noor in [1] is defined as

$$S_n(f;x)=\int_0^\infty W_n(x,t)f(t)=\sum_{v=1}^{\infty}s_{n,k}(x)\int_0^\infty f(t)b_{n,k}(t)dt+s_{n,0}(x)f(0), \tag{1.1}$$

where $W_n(x,t)=\sum_{k=1}^{\infty}s_{n,k}(x)b_{n,k}(t)+s_{n,0}(x)\delta(t)$, $\delta(t)$ being Dirac delta-function and

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\quad b_{n,k}(t)=\frac{1}{B(n+1,k)}\frac{t^{k-1}}{(1+t)^{n+k+1}},$$

are respectively Szász and Beta basis functions. In [1] Gupta and Noor studied some approximation properties for the operators defined in (1.1) and obtained the rate of point-wise convergence, a Voronovskaja type asymptotic formula and an error estimate in simultaneous approximation.

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In 1987, Lupaş introduced a q-analogue of the Bernstein operator and investigated its approximating and shape preserving properties. Ten years later, Phillips [2] proposed another generalization of the classical Bernstein polynomials based on q-integers. He obtained the rate of convergence and Voronovskaya-type asymptotic formula for these new Bernstein operators. An extension to q-calculus of Szász-Mirakyan operators was given by Aral [7] and established a Voronovskaja theorem related to q-derivatives for these operators.

In recent years, the application of q calculus is the most interesting areas of research in the approximation theory. Several authors have proposed the q analogues of different linear positive operators and studied their approximation behaviors. Gupta [5] introduced a q-analogue of usual Bernstein-Durrmeyer operators and established the rate of convergence of these operators. Gupta [6] proposed a generalization of the Baskakov operators based on q integers and estimated the rate of convergence in the weighted norm and some shape preserving properties.

Very recently in [9], Gupta introduced the q-analogue of Szász-Beta operators defined as

$$S_{n,q}(f(t);x)=\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s_{n,k}^q(x)\int_0^{\infty/A}p_{n,k}^q(t)f(qt)d_qt+E_q(-[n]_qx)f(0), \tag{1.2}$$

where

$$s_{n,k}^q(x)=\frac{([n]_qx)^k}{[k]_q!}E_q(-[n]_qq^kx), \quad p_{n,k}^q(t)=\frac{1}{B_q(n+1,k)}\frac{t^{k-1}}{(1+t)_q^{n+k+1}}. \tag{1.3}$$

While for $q=1$, these operators coincide with the Szász-Beta operators defined by (1.1).

First, we give some basic definitions and notations of q-calculus. All of the results can be found in [10, 11]. Throughout the present paper, we consider q as a real number such that $0<q<1$. For $n\in\mathbb{N}$. The q integer and q factorial are respectively defined as

$$[n]_q=\frac{1-q^n}{1-q}, \qquad [n]_q!=\begin{cases}[n]_q[n-1]_q\cdots[1]_q, & n\geq 1,\\ 1, & n=0.\end{cases}$$

The q-binomial coefficients are given by

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 0\leq k\leq n.$$

The q-Jackson integrals and the q-improper integrals are defined as (see [12])

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n, \quad a>0,$$

and

$$\int_0^{\infty/A}f(x)d_qx=(1-q)\sum_{n=-\infty}^{\infty}f\Big(\frac{q^n}{A}\Big)\frac{q^n}{A}, \quad A>0, \tag{1.4}$$