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C^p Condition and the Best Local Approximation

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Abstract. In this paper, we introduce a condition weaker than the L^p differentiability, which we call C^p condition. We prove that if a function satisfies this condition at a point, then there exists the best local approximation at that point. We also give a necessary and sufficient condition for that a function be L^p differentiable. In addition, we study the convexity of the set of cluster points of the net of best appoximations of f, $\{P_\epsilon(f)\}$ as $\epsilon\to 0$.

Key Words: Best L^p approximation, local approximation, L^p differentiability.

AMS Subject Classifications: 41A50, 41A10

1 Introduction

Let x_1, $a\in\mathbb{R}$, $a>0$, and let $\mathcal{L}$ be the space of equivalence class of Lebesgue measurable real functions defined on $I_a:=(x_1-a,x_1+a)$. For each Lebesgue measurable set $A\subset I_a$, with $|A|>0$, we consider the semi-norm on $\mathcal{L}$,

$$\|h\|_{p,A}:=\Big(|A|^{-1}\int_A |h(x)|^p dx\Big)^{1/p},\quad 1<p<\infty,$$

where $|A|$ denotes the measure of the set A. As usual, we denote by $L^p(I_a)$ the space of functions $h\in\mathcal{L}$ with $\|h\|_{p,I_a}<\infty$. If $0<\epsilon\le a$, $I_{-\epsilon}:=(x_1-\epsilon,x_1)$, $I_{+\epsilon}:=(x_1,x_1+\epsilon)$, we write $\|h\|_{p,\pm\epsilon}=\|h\|_{p,I_{\pm\epsilon}}$, and $\|h\|_{p,\epsilon}=\|h\|_{p,I_\epsilon}$. For a non negative integer s, we denote by Π^s the linear space of polynomials of degree at most s. Henceforward, we consider $n\in\mathbb{N}\cup\{0\}$. If $h\in L^p(I_a)$, it is well known that there exists a unique best $\|\cdot\|_{p,\epsilon}$-approximation of h from Π^n, say $P_\epsilon(h)$, i.e., $P_\epsilon(h)\in\Pi^n$ satisfies

$$\|h-P_\epsilon(h)\|_{p,\epsilon}\le\|h-P\|_{p,\epsilon}\quad\text{for all}\ \ P\in\Pi^n.$$

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$P_\epsilon(h)$ is the unique polynomial in Π^n, which verifies

$$\int_{I_\epsilon} |(h-P_\epsilon(h))(x)|^{p-1}\mathrm{sgn}((h-P_\epsilon(h))(x))(x-x_1)^j dx = 0, \quad 0\le j\le n, \tag{1.1}$$

see [2].

If $\lim_{\epsilon\to 0} P_\epsilon(h)$ exists, say $P_0(h)$, it is called the *best local approximation of h at x_1 from* Π^n (b.l.a.). In general, we shall also denote by $P_0(h)$ the set

$$\Big\{P\in\Pi^n : P=\lim_{k\to\infty} P_{\epsilon_k}(h) \text{ for some } \epsilon_k\downarrow 0\Big\}.$$

The problem of best local approximation was formally introduced and studied in a paper by Chui, Shisha and Smith [3]. However, the initiation of this could be dated back to results of J. L. Walsh [10], who proved that the Taylor polynomial of an analytic function h over a domain is the limit of the net of polynomial best approximations of a given degree, by shrinking the domain to a single point. Later, several authors studied the existence of the b.l.a. assuming a certain order of differentiability. In [8] and [12], this problem was considered when h is L^p differentiable. Recently, in [7] and [5] the authors proved the existence of the b.l.a. under weaker conditions, more precisely they assumed existence of lateral L^p derivatives of order n and L^p differentiability of order $n-1$. In [4] it was proved that if $p=2$ and h is differentiable up to order $n-1$, then $P_0(h)$ is either empty or convex. Later, in [11] using interpolation properties of the best approximation, the author extended this result for $1<p<\infty$. The main purpose of this paper is to give more general conditions on a function h so that there exists the b.l.a., and to study its connection with the L^p differentiability. Further, we study the convexity of $P_0(h)$. The following definition is motivated by the characterization (1.1).

Definition 1.1. We shall say that $f\in L^p(I_a)$ satisfies the C^p condition of order n at x_1, if there exists $Q\in\Pi^n$ such that

$$\int_{I_\epsilon} |(f-Q)(x)|^{p-1}\mathrm{sgn}((f-Q)(x))(x-x_1)^j dx = o(\epsilon^{n(p-1)+j+1}), \tag{1.2}$$

$0\le j\le n$, as $\epsilon\to 0$.

Analogously, we shall say that f satisfies the left (right) C^p condition of order n at x_1, if there exists $Q\in\Pi^n$ verifying (1.2) with $I_{-\epsilon}(I_{+\epsilon})$ instead of I_ϵ.

We denote with $c_n^p(x_1)$ the class of functions in $L^p(I_a)$ which satisfy the C^p condition of order n at x_1. We recall that a function $f\in L^p(I_a)$ is L^p *differentiable of order n at x_1* (i.e., $f\in t_n^p(x_1)$) if there exists $Q\in\Pi^n$ such that $\|f-Q\|_{p,\epsilon}=o(\epsilon^n)$. This concept was introduced by Calderón and Zygmund in [1]. Using the Hölder inequality, it is easy to see that $t_n^p(x_1)\subset c_n^p(x_1)$, moreover the inclusion is strict. In fact, if $h(x)=\sin(1/x)$, $x\neq 0$, then $h\in c_0^2(0)$, however a straightforward computation shows that $h\notin t_0^2(0)$. It immediately follows from Definition 1.1 that $c_n^p(x_1)$ satisfies: a) If $f\in c_n^p(x_1)$, then $f+P\in c_n^p(x_1)$ for