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Unique Common Fixed Point for a Family of Mappings with a Nonlinear Quasi-Contractive Type Condition in Metrically Convex Spaces

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Abstract. A class Φ of 5-dimensional functions was introduced and an existence and uniqueness of common fixed points for a family of non-self mappings satisfying a ϕ_i-quasi-contractive condition and a certain boundary condition was given on complete metrically convex metric spaces, and from which, more general unique common fixed point theorems were obtained. Our main results generalize and improve many same type common fixed point theorems in references.

Key Words: Metrically convex space, a class Φ of 5-dimensional functions, ϕ_i-quasi-contraction, common fixed point, complete.

AMS Subject Classifications: 47H25, 54H25

1 Introduction and preliminaries

There have appeared many fixed point theorems for a self-single-valued self mapping on a closed subset of a Banach space. However, in many applications, the map under considerations is not a self-mapping on a closed set. In 1976, Assad [1] gave a sufficient condition for such a single valued mapping to obtain a fixed point result by proving a fixed point theorem for a Kannan type mapping on a Banach space and putting certain boundary conditions on the mapping. Similar results for multi-valued mappings were respectively given by Assad [2] and Assad and Kirk [3]. Late, some authors obtained and generalized the same type results on complete metrically convex metric spaces [4–7]. The all above results were discussed under certain contractive conditions or certain boundary conditions. Recently, we further discussed the existence problems of unique common fixed points for the mappings satisfying certain contractive or quasi-contractive conditions with linear property, see [8–10]. In order to generalize and improve these

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results, we will discuss the the existence problems of unique common fixed points for the mappings satisfying certain ϕ-quasi-contractive conditions with nonlinear property and boundary conditions, and give more general results.

Definition 1.1 (see [4,5]). A metric space (X,d) is said to be metrically convex, if for any $x,y\in X$ with $x\neq y$, there exists $z\in X$ with $z\neq x$ and $z\neq y$ such that $d(x,z)+d(z,y)=d(x,y)$.

Lemma 1.1 (see [3,4]). *If K is a nonempty closed subset of a complete metrically convex space (X,d), then for any $x\in K$ and $y\notin K$, there exists $z\in\partial K$ such that $d(x,z)+d(z,y)=d(x,y)$.*

Definition 1.2. $\phi\in\Phi$ if and only if $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ is continuous, non-decreasing for every variable, and there exists $k\in[0,1/2)$ such that $\lambda(t):=\phi(t,t,t,t,2t)<kt$ for all $t>0$, where $\mathbb{R}^+=[0,\infty)$. Here, k is said to be the number of ϕ.

Example 1.1. Define $\phi_1,\phi_2:(\mathbb{R}^+)^5\to\mathbb{R}^+$ as follows

$$\phi_1(t_1,t_2,t_3,t_4,t_5)=k_1t_1+k_2t_2+k_3t_3+k_4t_4+k_5t_5,$$

where $k_1,k_2,k_3,k_4,k_5\in[0,1)$ satisfying $k_1+k_2+k_3+k_4+2k_5<1/2$.

$$\phi_2(t_1,t_2,t_3,t_4,t_5)=\max\{k_1t_1,k_2t_2,k_3t_3,k_4t_4,k_5t_5\},$$

where $k_1,k_2,k_3,k_4,k_5\in[0,1)$ and $\max\{k_1,k_2,k_3,k_4,2k_5\}<1/2$. Then $\phi_1,\phi_2\in\Phi$.

2 Common fixed point theorems

Next theorem is the main result in this paper.

Theorem 2.1. *Let K be a nonempty closed subset of a complete metrically convex space (X,d) with $\mathrm{int}K\neq\emptyset$, $\{T_i\}_{i\in\mathbb{N}}$ a family of self-mappings on X such that $K\subseteq T_i(X)$ for all $i\in\mathbb{N}$. Suppose that for any $x,y\in X$ and any $i,j\in\mathbb{N}$ with $i\leq j$,*

$$d(x,y)\leq\phi_i\big(d(T_ix,T_jy),d(x,T_ix),d(y,T_jy),d(x,T_jy),d(T_ix,y)\big), \tag{2.1}$$

where $\phi_i\in\Phi$ for all $i\in\mathbb{N}$. Suppose that $k=\sup_{i\in\mathbb{N}}k_i<1/2$, where k_i is the number of ϕ_i for all $i\in\mathbb{N}$. Furthermore, if $T_i(K^C)\cap\partial K=\emptyset$ for all $i\in\mathbb{N}$, and for any $i\in\mathbb{N}$ and $x\in K$, there exists $j\in\mathbb{N}$ such that $T_j(T_i(x))=x$, then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point in K.

Proof. Take any element $x_0\in K$. We will construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following manner. Since $K\subseteq T_1(X)$, there exists $x'_1\in X$ such that $x_0=T_1x'_1$. If $x'_1\in K$, then put $x_1=x'_1$. If $x'_1\notin K$, then there exists $x_1\in\partial K$ such that $d(x_0,x_1)+d(x_1,x'_1)=d(x_0,x'_1)$ by Lemma 1.1. Similarly $K\subseteq T_2(X)$ implies that there exists $x'_2\in X$ such that $x_1=T_2x'_2$. If $x'_2\in K$, then put$x_2=x'_2$; if $x'_2\notin K$, then there exists $x_2\in\partial K$ such that $d(x_1,x_2)+d(x_2,x'_2)=d(x_1,x'_2)$ by Lemma 1.1. Continuing in this way, we obtain $\{x_n\}$ and $\{x'_n\}$ with the following properties: