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L^p Bounds for the Commutators of Rough Singular Integrals Associated with Surfaces of Revolution

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Abstract. In this paper, we establish the $L^p(\mathbb{R}^{n+1})$ boundedness for the commutators of singular integrals associated to surfaces of revolution, $\{(t,\phi(|t|)):t\in\mathbb{R}^n\}$, with rough kernels $\Omega\in L(\log L)^2(\mathbb{S}^{n-1})$, if $\phi(|t|)=|t|$.

Key Words: Commutator, singular integral, surface of revolution, rough kernel.

AMS Subject Classifications: 42B20, 42B25

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1 Introduction

The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy,$$

where $\Omega\in L^1(S^{n-1})$ satisfies the following conditions:
(a) Ω is a homogeneous function of degree zero on $\mathbb{R}^n\backslash\{0\}$, i.e.,

$$\Omega(tx)=\Omega(x) \tag{1.1}$$

for any $t>0$ and $x\in\mathbb{R}^n\backslash\{0\}$.
(b) Ω has mean zero on S^{n-1}, the unit sphere in $\mathbb{R}^n$, i.e.,

$$\int_{S^{n-1}}\Omega(x')d\sigma(x')=0. \tag{1.2}$$

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The singular integral operator along the surface of revolution $\Gamma=\{(y,\phi(|y|));y\in R^n\}$ is defined by

$$T_{\phi,\Omega}f(x,x_{n+1})=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^n}f(x-y,x_{n+1}-\phi(|y|))dy.$$

In [9], Kim, Wainger, Wright and Ziesler proved the $L^p(R^{n+1})$ boundedness of $T_{\phi,\Omega}$ when $\phi\in C^2([0,\infty))$ is convex, increasing, $\phi(0)=0$ and $\Omega\in C^\infty(S^{n-1})$. In [2], Chen and Fan generalized the result in the case that Ω belongs to a block space introduced in [11]. In [10], Lu, Pan and Yang proved that when $\Omega\in H^1(S^{n-1})$ and ϕ satisfies some conditions, then $T_{\phi,\Omega}$ is bounded on $L^p(R^{n+1})$.

Grafakos and Stefanov [6] introduced a class of kernel functions $F_\alpha(S^{n-1})$ which are not contained in $H^1(S^{n-1})$. When $\Omega\in F_\alpha(S^{n-1})$, $T_{\phi,\Omega}$ was studied by many authors [3,12, 13].

For a function $b\in L_{loc}(\mathbb{R}^n)$, let A be a linear operator on some measurable function space. Then the commutator between A and b is defined by $[b,A]f(x):=b(x)Af(x)-A(bf)(x)$.

It has been proved by Hu [8] that $\Omega\in L(\log L)^2(\mathbb{S}^{n-1})$ is a sufficient condition for the commutator of singular integral to be bounded on $L^p(\mathbb{R}^n)$, which is defined by

$$[b,T_\Omega]f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))f(y)dy.$$

Recently, Chen and Ding [1] established the L^p boundedness of the commutator of singular integrals with the kernel condition $\Omega\in F_\alpha(S^{n-1})$.

The commutator of $T_{\phi,\Omega}$ is defined by

$$\begin{aligned}&[b,T_{\phi,\Omega}]f(x,x_{n+1})\\=&\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^n}(b(x,x_{n+1})-b(x-y,x_{n+1}-\phi(|y|)))f(x-y,x_{n+1}-\phi(|y|))dy.\end{aligned}$$

The main purpose of this paper is to investigate the L^p boundedness of $[b,T_{\phi,\Omega}]$ under the condition $\Omega\in L(\log L)^2(\mathbb{S}^{n-1})$, that is

$$\int_{\mathbb{S}^{n-1}}|\Omega(x')|\log^2(2+|\Omega(x')|)dx'<\infty.$$

The following theorem is the main result in this paper.

Theorem 1.1. *Let Ω satisfiy* (1.1) *and* (1.2), $b\in BMO(\mathbb{R}^{n+1})$, $\phi(|t|)=|t|$. *If $\Omega\in L(\log L)^2(\mathbb{S}^{n-1})$, then for all $1<p<\infty$, the commutator $[b,T_{\phi,\Omega}]$ is bounded on $L^p(\mathbb{R}^{n+1})$.*

It has been proved in [4] that the $L^p(\mathbb{R}^n)$ boundedness of the commutator of oscillatory singular integral operator T_ϕ defined by

$$[b,T_\phi]f(x)=\text{p.v.}\int_{\mathbb{R}^n}e^{i\phi(y)}\frac{\Omega(y)}{|y|^n}(b(x)-b(x-y))f(x-y)dy$$