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Hardy Spaces $H_L^p(\mathbb{R}^n)$ Associated with Higher-Order Schrödinger Type Operators

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Abstract. Let $L=L_0+V$ be the higher order Schrödiger type operator where L_0 is a homogeneous elliptic operator of order $2m$ in divergence form with bounded coefficients and V is a real measurable function as multiplication operator (e.g., including $(-\Delta)^m+V$ ($m\in\mathbb{N}$) as special examples). In this paper, assume that V satisfies a strongly subcritical form condition associated with L_0, the authors attempt to establish a theory of Hardy space $H_L^p(\mathbb{R}^n)$ $(0<p\le 1)$ associated with the higher order Schrödinger type operator L. Specifically, we first define the molecular Hardy space $H_L^p(\mathbb{R}^n)$ by the so-called (p,q,ε,M) molecule associated to L and then establish its characterizations by the area integral defined by the heat semigroup e^{-tL}.

Key Words: Higher order Schrödinger operator, off-diagonal estimates, H_L^p spaces, area integrals.

AMS Subject Classifications: 42B30, 42B25, 42B35, 35J15

1 Introduction

The real Hardy space $H^p(\mathbb{R}^n)$ which are extensively and deeply studied not only has a profound effect in the field of harmonic analysis, but also a major influence in several areas such as partial differential equations, analysis in several complex variables, analysis on symmetric spaces matrtingale theorem, and see, e.g., [15, 18, 27, 31], et al. Since the famous works by Stein and Weiss [28] and Fefferman and Stein [17] were published, real variable structure of Hardy spaces are explored, many more flexible real approaches to define Hardy spaces were discovered, including maximal operator M_Φ for some Schwarz function Φ with $\int\Phi=1$, Riesz transform, the atomic and molecule decompositions and the Littlewood-Paley theory et al. (see [6, 8, 17, 19, 26, 29]). In one word, these different

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characterizations on Hardy spaces provide very powerful insights and great interests in both theories and applications. Among them, there is an important point that one could connect these different characterizations with Laplacian and its corresponding Poisson semigroup $\{e^{-t\sqrt{-\Delta}}\}_{t\geq 0}$ and heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ (see [17,20,27], for example).

This paper is devoted to the study of Hardy spaces $H_L^p(\mathbb{R}^n)$ associated to higher order Schrödinger type operators $L=L_0+V$. In order to describe related background, let us first introduce the definition of L as follows. Let $m\geq 2$, $N\geq 1$ and L_0 be the following homogeneous elliptic operator of order $2m$ in divergence form

$$L_0 f=\sum_{|\alpha|=|\beta|=m}\partial^\alpha(a_{\alpha,\beta}\partial^\beta f), \tag{1.1}$$

which is interpreted by the following sesquilinear form

$$\mathcal{Q}_0(f,g):=\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^n}a_{\alpha,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}dx,\quad f,g\in W^{m,2}(\mathbb{R}^n), \tag{1.2}$$

where $W^{m,2}(\mathbb{R}^n)$ is the Sobolev spaces and for all $\alpha,\beta\in\mathbb{N}^n$ with $|\alpha|=|\beta|=m$, $a_{\alpha,\beta}\in L^\infty(\mathbb{R}^n,\mathbb{C})$ satisfying $a_{\alpha,\beta}=\overline{a_{\beta,\alpha}}$ and the strong Gårding inequality

$$\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^n}a_{\alpha,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha f(x)}dx\geq\rho\|\nabla^m f\|^2_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

for some $\rho>0$ independent of $f\in W^{m,2}(\mathbb{R}^n)$. Assume that $V\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the strongly subcritical condition, that is,

$$\int_{\mathbb{R}^n}V_-|f|^2dx\leq\mu\Big(\mathcal{Q}_0(f,f)+\int_{\mathbb{R}^n}V_+|f|^2dx\Big), \tag{1.4}$$

for some $\mu\in[0,1)$ and all $f\in W^{m,2}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}V_+|f|^2dx<\infty$, where $V_\pm(x):=\max\{0,\pm V(x)\}$ denote the positive and negative parts of V, respectively. It follows from the KLMN theorem (e.g., see Kato [25, pp. 338]) that L is well-defined as a nonnegative self-adjoint operator associated to the sesquilinear form

$$\mathcal{Q}(f,g):=\mathcal{Q}_0(f,g)+\int_{\mathbb{R}^n}V_+f\overline{g}dx-\int_{\mathbb{R}^n}V_-f\overline{g}dx \tag{1.5}$$

on the domain $D(\mathcal{Q})=\{f\in W^{m,2}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}V_+|f|^2dx<\infty\}$.

We denote the set of all higher order Schrödinger type operator L associated to $\mathcal{Q}$ defined in (1.5) by

$$\mathscr{A}_m(\rho,\Lambda,\mu)=\{L:L=L_0+V\},$$

where $\Lambda=\sup|a_{\alpha,\beta}|$, ρ is the constant in (1.3) and $\mu\in[0,1)$ is the smallest constant such that (1.4) holds. Let

$$\mathscr{A}_m=\bigcup_{\rho,\Lambda,\mu}\mathscr{A}_m(\rho,\Lambda,\mu).$$