Weak Type Weighted Inequalities for the Commutators of the Multilinear Calderón-Zygmund Operators

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Abstract. For the commutators of multilinear Calderón-Zygmund singular integral operators with BMO functions, the weak type weighted norm inequalities with respect to $A_p$ weights are obtained.

Key Words: Commutator, multilinear Calderón-Zygmund operator, multilinear maximal function, weight.

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1 Introduction

The multilinear Calderón-Zygmund theory originated in the works of Coifman and Meyer [2,3]. Later on the topic was retaken by several authors; including Christ and Journé [1] Kenig and Stein [7], and Grafakos and Torres [5,6]. We first recall the definitions of multilinear Calderón-Zygmund singular integral operators and commutators.

Let $T$ be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T: S(R^n) \times \cdots \times S(R^n) \rightarrow S'(R^n).$$

We say that $T$ is an multilinear Calderón-Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^q$, where $1/q = 1/q_1 +$
\( \cdots + 1 / q_m \), and if there exists a function \( K \), defined off the diagonal \( x = y_1 = \cdots = y_m \) in \((\mathbb{R}^n)^{m+1}\), for \( \tilde{f} = (f_1, \cdots, f_m) \), satisfying

\[
T(\tilde{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) d\vec{y}
\]

for all \( x \notin \bigcap_{j=1}^m \text{supp} f_j \), where \( d\vec{y} = dy_1 \cdots dy_m \) and \( \vec{y} = (y_1, \cdots, y_m) \);

\[
|K(y_0, y_1, \cdots, y_m)| \leq \frac{A}{(\sum_{k,j=0}^m |y_k - y_j|)^{mn}} \tag{1.1}
\]

and

\[
|K(y_0, \cdots, y_j, \cdots, y_m) - K(y_0, \cdots, y_j', \cdots, y_m)| \leq \frac{A |y_j - y_j'|^{\gamma}}{\left(\sum_{k,j=0}^m |y_k - y_j|\right)^{mn+\gamma}} \tag{1.2}
\]

for some \( \gamma > 0 \) and all \( 0 \leq j \leq m \), whenever \( |y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k| \).

For a vector \( \vec{b} = (b_1, \cdots, b_m) \) of locally integrable functions, we define the commutator of multilinear singular integral operators

\[
T_{\vec{b}}(\tilde{f})(x) = \sum_{j=1}^m T_{b_j}^j(\tilde{f})(x) = \sum_{j=1}^m \int_{\mathbb{R}^n} (b_j(x) - b_j(y_j)) K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) d\vec{y}.
\]

Recently, Lerner, Ombrosi, Pérez and Trujillo-González [8] defined a new multilinear maximal function associated to the \( m \)-linear Calderón-Zygmund operators.

Let \( 1 \leq p_1, \cdots, p_m < \infty \), we will write \( p \) for the number given by \( 1 / p = 1 / p_1 + \cdots + 1 / p_m \), and \( \vec{p} = (p_1, \cdots, p_m) \). Given \( \vec{w} = (w_1, \cdots, w_m) \), \( w_j \) are nonnegative locally integrable functions on \( \mathbb{R}^n \), \( j = 1, \cdots, m \), set \( \vec{w} = \prod_{j=1}^m w_j^{p_j/p_j} \). We say that \( \vec{w} \) satisfies the \( A_{\vec{p}} \) condition if

\[
\sup_Q \left( \frac{1}{|Q|^q} \int_Q w_j \right)^{1/p_j} \prod_{j=1}^m \left( \frac{1}{|Q|^q} \int_Q w_j^{1-p_j} \right)^{1/p_j} < \infty.
\]

When \( p_j = 1 \), \( \left( \frac{1}{|Q|^q} \int_Q w_j^{1-p_j} \right)^{1/p_j} \) is understood as \((\inf_Q w_j)^{-1}\). For \( m = 1 \), \( A_{\vec{p}} \) is the Muckenhoupt weight class \( A_p \). We denote \( A_{\infty} = \bigcup_{p>1} A_p \).

Lerner, Ombrosi, Pérez and Trujillo-González [8] obtained the following weighted inequalities for the commutators of the multilinear singular integral operators.