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On a Pair of Operator Series Expansions Implying a Variety of Summation Formulas

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Abstract. With the aid of Mullin-Rota's substitution rule, we show that the Sheffer-type differential operators together with the delta operators Δ and D could be used to construct a pair of expansion formulas that imply a wide variety of summation formulas in the discrete analysis and combinatorics. A convergence theorem is established for a fruitful source formula that implies more than 20 noted classical fomulas and identities as consequences. Numerous new formulas are also presented as illustrative examples. Finally, it is shown that a kind of lifting process can be used to produce certain chains of (∞^m) degree formulas for $m\geq 3$ with $m\equiv 1$ (mod 2) and $m\equiv 1$ (mod 3), respectively.

Key Words: Delta operator, Sheffer-type operator, (∞^m) degree formula, triplet, lifting process.

AMS Subject Classifications: 12E10, 13F25, 16S32, 65B10

1 Introduction and preliminaries

Throughout this paper the theory of formal power series and of differential operators will be utilized. **R**, **C**, **N** and **Z** denote, respectively, the sets of real numbers, complex numbers, natural numbers (including 0) and rational integers. Generally, we will use $A(t)$, $g(t)$, $f(t)$, $\varphi(t)$, etc. to denote either the formal power series (fps) or the infinitely differentiable functions (members of C^∞) defined in R or **C**.

We will make use of the ordinary operators Δ (difference), D (differentiation) and E (shift operator) which are defined by the relations respectively

$$\Delta f(t)=f(t+1)-f(t),\quad Df(t)=\frac{d}{\alpha t}f(t),\quad Ef(t)=f(t+1).$$

Consequently they satisfy some simple symbolic relations such as

$$E=1+\Delta,\quad E=e^D,\quad \Delta=e^D-1,\quad D=\log(1+\Delta),$$

*Corresponding author. *Email address:* xulizhi63@hotmail.com (L. C. Hsu) where 1 serves as an identity operator such that $1f(x)=f(x)$. Also, we define $E^\alpha f(t)=f(t+\alpha)$ for $\alpha\in\mathbf{R}$ or $\mathbf{C}$ with $E^0=D^0=\Delta^0=1$. Hereafter, we always assume that all the operators are acting at the variable t of a function or a fps.

Generally, an operator T is called a shift-invariant operator (see [8]) if $TE^\alpha=E^\alpha T$ for every $\alpha\in\mathbf{R}$. Moreover, if in addition, $Tt\neq 0$ (a non-zero constant), then T is called a delta operator. Obviously, both Δ and D are delta operators.

We shall frequently utilize a general prposition due to Mullin and Rota as stated in what follows (cf. [8]).

Let Q be a delta operator, and let Γ be the ring of formal power series in t, over the same number field (**R** or **C**). Then there exists an isomorphism from Γ into the ring Σ of shift-invariant operators, which carries

$$f(t)=\sum_{k\geq 0}\frac{a_k}{k!}t^k \quad \text{into} \quad f(Q)\equiv f(t,Q)=\sum_{k\geq 0}\frac{a_k}{k!}Q^k.$$

The above assertion is called Mullin-Rota's substitution rule.

For the fps $f(t)$ given above, note that $D^kf(t)\equiv f^{(k)}(t)$ means a kth order formal derivative of $f(t)$, so that $D^kf(0)\equiv f^{(k)}(0)=a_k$, and that $f(t)$ can also be written as a formal Taylor series.

Also, we shall need three basic concepts as given by the following definitions.

Definition 1.1. For any given fps $A(t)$ and $g(t)$ such that $A(0)=1$, $g(0)=0$ and $g'(0)=Dg(0)\neq 0$, the polynomials $p_k(z)$ $(k\in\mathbf{N})$, defined by the generating function (GF)

$$A(t)e^{zg(t)}=\sum_{k\geq 0}p_k(z)t^k \tag{1.1}$$

are called the Sheffer-type polynomials, where $p_0(z)=1$. More explicitly, we may denote

$$p_k(z)\equiv p_k(z,A(t),g(t))=[t^k]A(t)e^{zg(t)}, \tag{1.2}$$

where $[t^k]$ is the so-called extracting coefficient operator.

Accordingly, $p_k(D)$ with $D\equiv d/dt$ is called the Sheffer-type differcntial operator of degree k. In particular, $p_0(D)\equiv 1$ is the identity operator.

Definition 1.2. Any expansion formula or a summation formula in the theory of formal power series as well as in the computational analysis is called an (∞^m) degree formula if it consists of m arbitrary functions that could be chosen in infinitely many ways, where m is called the freedom degree of the formula.

For example, the expansion (1.1) is an (∞^2) degree formula.

Definition 1.3. For $A(t)$ and $g(t)$ as given in Definition 1.1, the numbers d_{kj} as defined by (cf. [6,7,12] etc).

$$d_{kj}=[t^k]A(t)(g(t))^j, \quad 0\leq j\leq k\in\mathbf{N}, \tag{1.3}$$

are said to form a Riordan array (d_{kj}) which may be denoted by $(A(t),g(t))$.