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Variable Exponent Herz Type Hardy Spaces and Their Applications

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Abstract. In this paper, the authors introduce certain Herz type Hardy spaces with variable exponents and establish the characterizations of these spaces in terms of atomic and molecular decompositions. Using these decompositions, the authors obtain the boundedness of some singular integral operators on the Herz type Hardy spaces with variable exponents.

Key Words: Hardy space, Herz space, variable exponent, maximal function, atom, molecule.

AMS Subject Classifications: 42B35, 42B25, 42B20

1 Introduction

In 1991, Kováčik and Rákosník introduced basic properties of variable Lebesgue and Sobolev spaces in [23]. After that many spaces with variable exponents appeared, for example: Bessel potential spaces with variable exponent, Besov and Triebel-Lizorkin spaces with variable exponents, Morrey spaces with variable exponents, Hardy spaces with variable exponent and so on, see [2,3,9,11,15,17,21,22,31,41] and reference therein. Indeed, the atomic, molecular and wavelet decompositions of variable exponent Besov and Triebel-Lizorkin spaces were given in [3,9,21,22,42]. The duality and reflexivity of spaces $B^s_{p(\cdot),q}$ and $F^s_{p(\cdot),q}$ were discussed in [33]. The atomic and molecular decompositions of Hardy spaces with variable exponent and their applications for the boundedness of singular integral operators were obtained in [31,34]. Variable exponent spaces have many applications, for instance in differential equations, see the article [16].

In parallel to the above, there are other type function spaces, Herz type spaces, have attracted many authors' interests for last three decades. Actually, many properties of classic Lebesgue spaces have been generalized to Herz type spaces. We outline some recent results we have concerned. In 2010, Izuki proved the boundedness of sublinear operators

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on Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}$ and $K^{\alpha,q}_{p(\cdot)}$ in [19]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ in [1]. Shi and the second author in [35] considered Herz type Besov and Triebel-Lizorkin spaces with one variable exponent. Then the authors in [10] established the boundedness of vector-valued Hardy-Littlewood maximal operator in spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ and gave characterizations of Herz type Besov and Triebel-Lizorkin spaces with variable exponents by maximal functions. In [40], Hongbin Wang and Zongguang Liu introduced a certain Herz type Hardy spaces with variable exponent which is a generalization of classical Herz type Hardy spaces, for the latter, see the monograph [30] by Shanzhen Lu, Dachun Yang and Guoen Hu. For more information about the generalizations, see [18, 24–26, 29, 38, 43, 44].

Inspired by the previous articles, we would like to declare that the goal of this paper is to introduce new Herz type Hardy spaces with variable exponents and give their applications. The structure of the paper is as follows. In the rest of the section we shall recall some definitions and notations. In Section 2, we shall define the Herz type Hardy spaces with variable exponents $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$, and give their atomic characterizations. In Section 3, we shall present the molecular characterizations of $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$.

Throughout this paper, $|S|$ denotes the Lebesgue measure and χ_S the characteristic function for a measurable set $S\subset\mathbb{R}^n$. For a multi-index $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$, we denote $|\beta|=\beta_1+\beta_2+\cdots+\beta_n$. We also use the notation $a\lesssim b$ if there exist a constant $C>0$ such that $a\le Cb$. If $a\lesssim b$ and $b\lesssim a$ we will write $a\approx b$. Finally we claim that C is always a positive constant but it may change from line to line.

Definition 1.1. Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. Denote

$$L^{p(\cdot)}(\mathbb{R}^n):=\{f \text{ is measurable on } \mathbb{R}^n:\rho_{p(\cdot)}(f/\lambda)<\infty \text{ for some constant } \lambda>0\},$$

where $\rho_{p(\cdot)}(f):=\int_{\mathbb{R}^n}|f(x)|^{p(x)}dx$, and

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\{\lambda>0:\rho_{p(\cdot)}(f/\lambda)\le 1\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $L^1_{\rm loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f\in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x):=\sup_{r>0}r^{-n}\int_{B(x,r)}|f(y)|dy,\ \forall x\in\mathbb{R}^n,$$

where $B(x,r):=\{y\in\mathbb{R}^n:|x-y|<r\}$. We also use the following notation: $p_-:=\operatorname{essinf}\{p(x):x\in\mathbb{R}^n\}$ and $p_+:=\operatorname{esssup}\{p(x):x\in\mathbb{R}^n\}$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_->1$