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The Negative Spectrum of Schrödinger Operators with Fractal Potentials

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Abstract. Let $\Gamma\subset\mathbb{R}^2$ be a regular anisotropic fractal. We discuss the problem of the negative spectrum for the Schrödinger operators associated with the formal expression

$$H_\beta = id - \Delta + \beta tr_b^\Gamma, \quad \beta\in\mathbb{R},$$

acting in the anisotropic Sobolev space $W_2^{1,\alpha}(\mathbb{R}^2)$, where Δ is the Dirichlet Laplanian in $\mathbb{R}^2$ and tr_b^Γ is a fractal potential (distribution) supported by Γ.

Key Words: Anisotropic function space, anisotropic fractal, Schrödinger operators, negative eigenvalues.

AMS Subject Classifications: 47G30, 58J50

1 Introduction

Let $a(x,D)$ be a positive-definite, self-adjoint pseudodifferential operator in $L_2(\mathbb{R}^2)$, $W_1^2(\mathbb{R}^2)$ the classical Sobolev space. In general, we studied the operators of the type

$$H_\beta = a(x,D) + \beta V, \quad \beta\in\mathbb{R}, \tag{1.1}$$

where V is a generalized potential given by a distribution supported by a set $\Gamma\subset\mathbb{R}^2$ with Lebesgue measure $|\Gamma|=0$. Of interest is the number of eigenvalues, counted with respect to their multiplicities, in $(-\infty,0]$ in dependence on $\beta\to\infty$. Recall that $\sharp M$ denotes the number of a finite set M. Let $\sigma(H_\beta)$ be the spectrum of self-adjoint operator H_β in $L_2(\mathbb{R}^2)$, then

$$\sharp\big\{\sigma(H_\beta)\cap(-\infty,0]\big\} \le \sharp\big\{k\in\mathbb{N}: \sqrt{2}\beta e_k\big(a(x,D)^{-1}V\big)\ge 1\big\}. \tag{1.2}$$

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The study of $\sharp\{\sigma(H_\beta)\cap(-\infty,0]\}$ is sometimes called a problem of the negative spectrum.

The interest of the problem comes from quantum mechanics with the typical hydrogen-like operator

$$H=-\frac{h^2}{2m}\Delta+V(x), \tag{1.3}$$

where h is the Planck's constant, m is mass of the electron and $V(x)$ is the potential. Asking for (negative) eigenvalues one can divide H by $h^2/2m$ and obtains

$$H_\beta=-\Delta+\beta V(x), \tag{1.4}$$

where β is proportional to h^2. Of interest is the semi-classical limit where $h\to 0$, or, $\beta\to\infty$, hence the behaviour of the cardinal number in dependence on β.

In slight modification of (1.3), we are interested in the negative spectrum of

$$H_\beta=id-\Delta+\beta tr_b^\Gamma, \tag{1.5}$$

where tr_b^Γ is the interpretation of btr_Γ and $b\in L_r(\Gamma)$ is real with $0\le 1/r<1$, tr_Γ is the trace operator. Let N_β be the number of negative eigenvalues. If Γ is an (isotropic) d-set, we obtained satisfactory results, $N_\beta\sim\beta^1$.

For anisotropic fractals in $\mathbb{R}^2$, an anisotropic d-set Γ having deviation, $0\le a\le 1$ is, roughly speaking, a compact set which can be covered for any $j\in\mathbb{N}_0$ with $N_j\sim 2^{jd}$ rectangles $\{R_l^j: l=1,\cdots,N_j\}$, $c2^{-2j}\le vol R_l^j\le 2^{-2j}$, for some c, $0<c<1$, independent of j and l, having the sides parallel to the axes of coordinates and side lengths $2^{-a_1^{j,l}}$, $2^{-a_2^{j,l}}$ satisfying

$$(1-a)j\le a_1^{j,l}\le a_2^{j,l}\le(1+a)j$$

for any $l=1,\cdots,N_j$. In this case, one has for N_β only two side estimates of type

$$C_1\beta^{\omega_1}\le N_\beta\le C_2\beta^{\omega_2} \tag{1.6}$$

for appropriate positive numbers ω_1 and ω_2 with $\omega_1\le 1\le\omega_2$.

The aim of this paper is to show that sharp estimate of (1.6), restricting ourselves to the regular anisotropic fractals and the Schrödinger operators acting in the anisotropic Sobolev space $W_2^{1,\alpha}(\mathbb{R}^2)$. We will show that

$$N_\beta\sim\beta^1. \tag{1.7}$$

In Section 2 we present basic facts concerning anisotropic function spaces, regular anisotropic fractals and some preparatory facts for the proofs. The main results, containing the precise proof of (1.6), are presented in Section 3.