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Spectral Self-Affine Measures on the Generalized Three Sierpinski Gasket

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Abstract. The self-affine measure $\mu_{M,D}$ associated with an iterated function system $\{\phi_d(x)=M^{-1}(x+d)\}_{d\in D}$ is uniquely determined. It only depends upon an expanding matrix M and a finite digit set D. In the present paper we give some sufficient conditions for finite and infinite families of orthogonal exponentials. Such research is necessary to further understanding the non-spectral and spectral of $\mu_{M,D}$. As an application, we show that the $L^2(\mu_{M,D})$ space has infinite families of orthogonal exponentials on the generalized three Sierpinski gasket. We then consider the spectra of a class of self-affine measures which extends several known conclusions in a simple manner.

Key Words: Compatible pair, orthogonal exponentials, spectral measure.

AMS Subject Classifications: 28A80, 42C05

1 Introduction

We consider in the paper the spectral and non-spectral problems of self-affine measures on the generalized three Sierpinski gasket. The problems originate from the discovery of Jorgensen and Pedersen [2] that some fractal measures also admit exponential orthogonal bases, but some do not. It generates a lot of interest in understanding what kind of measures are spectral measure. For those measures failing to have exponential orthogonal bases, one of the main problems is to estimate the number of orthogonal exponentials in $L^2(\mu)$.

The finite or infinite $\mu_{M,D}$-orthogonal exponential plays an important role in the study of non-spectral or spectral of self-affine measure $\mu_{M,D}$. We consider in the paper the questions of when the $L^2(\mu_{M,D})$ space has finite and infinite families of orthogonal exponentials.

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Let $M\in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, one with all eigenvalues $|\lambda_i(M)|>1$ and $D\subseteq\mathbb{Z}^n$ being a finite subset of cardinality $|D|$. Associated with iterated function system $(IFS)\{\phi_d(x)=M^{-1}(x+d)\}_{d\in D}$, there exists a unique probability measure $\mu:=\mu_{M,D}$ satisfying the self-affine identity (see [12])

$$\mu=\frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1}. \tag{1.1}$$

Such μ is supported on $T(M,D)$ and is called a self-affine measure. A more explicit expression of T is given by the following radix expansion

$$T(M,D)=\Big\{\sum_{j=1}^{\infty}M^{-j}d_j:\ d_j\in D\Big\}. \tag{1.2}$$

Let $S\subseteq\mathbb{Z}^n$ be a finite subset with $|S|=|D|$, corresponding to the dual IFS $\{\psi_s(x)=M^*x+s\}_{s\in S}$, we use $\Lambda(M^*,S)$ to denote the expansive orbit of 0 under $\{\psi_s(x)\}_{s\in S}$, that is

$$\Lambda(M^*,S):=\Big\{\sum_{j=0}^{k-1}M^{*j}s_j:\ k\geq 1\text{ and } s_j\in S\Big\}, \tag{1.3}$$

where M^* is the conjugate transpose matrix of M.

Recall that for a probability measure $\mu_{M,D}$ of compact support on $\mathbb{R}^n$, we call $\mu_{M,D}$ a spectral measure if there exists a discrete set $\Lambda\subseteq\mathbb{R}^n$ such that $E_\Lambda:=\{e^{2\pi i\langle\lambda,x\rangle}:\ \lambda\in\Lambda\}$ forms an orthogonal basis for $L^2(\mu_{M,D})$. The set Λ is then called a spectrum for $\mu_{M,D}$.

Spectral measure is a natural generalization of spectral set introduced by Fuglede [18] whose famous conjecture and its related problems have received much attention in recent years (see [13, 14]). The spectral self-affine measure problem at the present day has been studied in the papers [1–4] (see also [15, 16] for the main goal).

It is known that the spectral problem on self-affine measures consists of the following two classes:

(I) There are natural infinite families of orthogonal exponentials.

(II) One of infinite families forms an orthogonal basis in $L^2(\mu_{M,D})$. The questions concerning this class can be found in [5–11].

Recall that the self-affine measure $\mu_{M,D}$ corresponding to

$$M=\begin{bmatrix} p_1 & 0 & 0\\ 0 & p_2 & 0\\ 0 & 0 & p_3\end{bmatrix},\quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},$$

is supported on the three-dimensional Sierpinski gasket $T(M,D)$. Because of the efforts of Jorgensen and Pedersen (see [4, Theorem 5.1(iii)]) and Li (see [11, Theorem 1.1(i)] and [19, Theorem 2.1]), the spectral self-affine measures are known only in the case when $p_j\in 2\mathbb{Z}\setminus\{0\}$ for $j=1,2,3$, then $\mu_{M,D}$ is a spectral measure. The general case for the spectrality of the self-affine measure $\mu_{M,D}$ is not known.