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On the Approximation of an Analytic Function Represented by Laplace-Stieltjes Transformation

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Abstract. In the present paper, we have considered the approximation of analytic functions represented by Laplace-Stieltjes transformations using sequence of definite integrals. We have characterized their order and type in terms of the rate of decrease of $E_n(F,\beta)$ where $E_n(F,\beta)$ is the error in approximating of the function $F(s)$ by definite integral polynomials in the half plane $Res \leq \beta < \alpha$.

Key Words: Laplace-Stieltjes transformation, analytic function, order, type, approximation error.
AMS Subject Classifications: 30D15, 32A15

1 Introduction

Consider the Laplace-Stieltjes transformation defined by

$$G(s)=\int_0^\infty \exp(-sx)d\alpha(x), \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0,X]$, $(0<X<+\infty)$, $s=\sigma+it$, σ and t are real variables. We choose a monotonic increasing sequence of real numbers $\{\lambda_n\}$ satisfying the following conditions:

$$0=\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n\uparrow+\infty, \tag{1.2a}$$

$$\lim_{n\to\infty}\sup(\lambda_{n+1}-\lambda_n)=\delta<+\infty, \quad \sup\frac{n}{\lambda_n}=D<+\infty. \tag{1.2b}$$

We put

$$K_n^*=\sup_{\lambda_n<x\leq\lambda_{n+1},-\infty<t<+\infty}\Big|\int_{\lambda_n}^x e^{-ity}d\alpha(y)\Big|.$$

In [7], Yu Jiarong obtained the following Valiron-Knopp-Bohr formula:

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Theorem 1.1. *Suppose that the Laplace-Stieltjes transformation* (1.1) *satisfies*

$$\lim_{n\to\infty}\sup(\lambda_{n+1}-\lambda_n)<+\infty,\quad \lim_{n\to\infty}\sup\frac{\ln n}{\lambda_n}<+\infty,$$

and σ_μ^G *denotes the abscissa of uniform convergence of* (1.1). *Then*

$$\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}\le\sigma_\mu^G\le\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}+\lim_{n\to\infty}\sup\frac{\ln n}{\lambda_n}. \tag{1.3}$$

Suppose that

$$\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}=0. \tag{1.4}$$

If $D=0$ then by (1.2b), (1.3) and (1.4), it follows that $\sigma_\mu^G=0$ and $G(s)$ is analytic in the right half plane $\sigma>0$. Kong and Yang [8] considered the Laplace-Stieltjes transformations given by (1.1) converging uniformly in the whole complex plane $Re(s)>-\infty$ and studied their growth properties.

In 2012, Luo Xi and Kong Yinying [5] defined Laplace-Stieltjes transformations in a different manner by taking positive exponents in the integral (1.1). Thus they defined Laplace-Stieltjes transformations as given below:

$$F(s)=\int_0^{+\infty}\exp(sy)d\alpha(y),(s=\sigma+it), \tag{1.5}$$

where $\alpha(y)$ satisfies the same conditions as stated earlier and the sequence $\{\lambda_n\}$ satisfies both conditions stated in (1.2b). We put

$$A_n^*=\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\Big|\int_{\lambda_n}^x e^{ity}d\alpha(y)\Big|.$$

A result similar to that of Theorem A can be proved easily for the integral (1.5) also. If the integral in (1.5) converges absolutely in the half plane $Res<\alpha$ $(-\infty<\alpha<\infty)$, then it represents an analytic function in $Res<\alpha$ and since (1.2a) holds we have

$$\lim_{n\to\infty}\inf\frac{\ln(A_n^*)^{-1}}{\lambda_n}=\alpha. \tag{1.6}$$

Definition 1.1. We define maximum modulus, the maximum term and the central index of the function $F(s)$ defined by (1.5) as

$$M(\sigma,F)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|,$$
$$M_\mu(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\Big|\int_0^x e^{sy}d\alpha(y)\Big|,s=\sigma+it,\quad \sigma<\alpha,$$
$$\mu(\sigma,F)=\max_{1\le n<N}\{A_n^*e^{\lambda_n\sigma}\},\quad \sigma<\alpha,$$
$$N(\sigma,F)=\max\{n;\mu(\sigma,F)=A_n^*e^{\lambda_n\sigma}\},$$