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Hardy Type Estimates for Riesz Transforms Associated with Schrödinger Operators on the Heisenberg Group

Yu Liu and Guobin Tang*

School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

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Abstract. Let $\mathbb{H}^n$ be the Heisenberg group and $Q=2n+2$ be its homogeneous dimension. In this paper, we consider the Schrödinger operator $-\Delta_{\mathbb{H}^n}+V$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and V is the nonnegative potential belonging to the reverse Hölder class B_{q_1} for $q_1 \geq Q/2$. We show that the operators $T_1 = V(-\Delta_{\mathbb{H}^n}+V)^{-1}$ and $T_2 = V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}$ are both bounded from $H^1_L(\mathbb{H}^n)$ into $L^1(\mathbb{H}^n)$. Our results are also valid on the stratified Lie group.

Key Words: Heisenberg group, stratified Lie group, reverse Hölder class, Riesz transform, Schrödinger operator.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

Let $L=-\Delta_{\mathbb{H}^n}+V$ be a Schrödinger operator, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and V the nonnegative potential belonging to the reverse Hölder class B_{q_1} for some $q_1 \geq Q/2$ and $Q>5$. In this paper we consider the Riesz transforms associated with the Schrödinger operator L

$$T_1=V(-\Delta_{\mathbb{H}^n}+V)^{-1}, \quad T_2=V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}, \quad T_3=\nabla_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}.$$

We are interested in the Hardy type estimates for the Riesz transform T_i, i=1,2,3. In recent years, some problems related to Schrödinger operators and Schrödinger type operators on the Heisenberg group and other nilpotent Lie group have been investigated by a number of scholars (see [2,3,5–10,12]). Among these papers the core problem is the research of estimates for Riesz transforms associated with the Schrödinger operator L. As we know, C. C. Lin, H. P. Liu and Y. Liu have proved that the operator $T_3=\nabla_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}$ is

*Corresponding author. *Email addresses:* `liuyu75@pku.org.cn` (Y. Liu), `tgb0405@126.com` (G. B. Tang)

bounded from $H^1_L(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$ in [5]. In this paper we will show that the other two operators T_1 and T_2 are also bounded from $H^1_L(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$. At the last section, we simply state the results on the stratified Lie group.

In what follows we recall some basic facts for the Heisenberg group $\mathbb{H}^n$ (cf. [11]). The Heisenberg group $\mathbb{H}^n$ is a lie group with the underlying manifold $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, and the multiplication

$$(x,y,t)(x',y',t')=(x+x',y+y',t+t'+2x'y-2xy').$$

A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},Y_{j=}\frac{\partial}{\partial y_j}+2x_j\frac{\partial}{\partial t},T=\frac{\partial}{\partial t},\quad j=1,2,\cdots,n.$$

All non-trivial commutation relations are given by $[X_j,Y_j]=-4T$, $j=1,2,\cdots,n$. Then the sub-Laplacian $\Delta_{\mathbb{H}^n}$ is defined by $\Delta_{\mathbb{H}^n}=\sum_{j=1}^n(X_j{}^2+Y_j{}^2)$ and the gradient operator $\nabla_{\mathbb{H}^n}$ is defined by

$$\nabla_{\mathbb{H}^n}=(X_1,\cdots,X_n,Y_1,\cdots,Y_n).$$

The dilations on $\mathbb{H}^n$ have the form $\delta_\lambda(x,y,t)=(\lambda x,\lambda y,\lambda^2 t),\lambda>0$. The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. We denote the measure of any measurable set E by $|E|$. Then $|\delta_\lambda E|=\lambda^Q|E|$, where $Q=2n+2$ is called the homogeneous dimension of $\mathbb{H}^n$.

We define a homogeneous norm function on $\mathbb{H}^n$ by

$$|g|=((|x|^2+|y|^2)^2+|t|^2)^{\frac{1}{4}},\quad g=(x,y,t)\in\mathbb{H}^n.$$

This norm satisfies the triangular inequality and leads to a left-invariant distant function $d(g,h)=|g^{-1}h|$. Then the ball of radius r centered at g is given by

$$B(g,r)=\{h\in\mathbb{H}^n:|g^{-1}h|<r\}.$$

The ball $B(g,r)$ is the left translation by g of $B(0,r)$ and we have $|B(g,r)|=\alpha_1 r^Q$, where $\alpha_1=|B(0,1)|$, but it is not important for us.

A nonnegative locally L^q integrable function V on $\mathbb{H}^n$ is said to belong to B_q $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(g)^q dg\Big)^{\frac{1}{q}}\le\frac{C}{|B|}\int_B V(g)dg$$

holds for every ball B in $\mathbb{H}^n$.

It is obvious that $B_{q_2}\subset B_{q_1}$ where $q_2>q_1$. From [3] we know that the B_q class has a property of "self improvement"; that is, if $V\in B_q$, then $V\in B_{q+\varepsilon}$ for some $\varepsilon>0$.

Assume that $V\in B_{q_1}$ for some $q_1>Q/2$. The definition of the auxiliary function $m(g,V)$ is given as follows.