Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 2 (2016), pp. 103-121

DOI: 10.4208/ata.2016.v32.n2.1

Multilinear Fractional Integrals and Commutators on Generalized Herz Spaces

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Received 22 October 2015; Accepted (in revised version) 22 January 2016

Abstract. Suppose $\vec{b}=(b_1,\cdots,b_m)\in(BMO)^m$, $I^{\Pi b}_{\alpha,m}$ is the iterated commutator of $\vec{b}$ and the m-linear multilinear fractional integral operator $I_{\alpha,m}$. The purpose of this paper is to discuss the boundedness properties of $I_{\alpha,m}$ and $I^{\Pi b}_{\alpha,m}$ on generalized Herz spaces with general Muckenhoupt weights.

Key Words: Multilinear fractional integral, generalized Herz space, commutator, Muckenhoupt weight.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $(\mathbb{R}^n)^m=\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ be the m-fold product space ($m\in\mathbb{N}$), and let $\vec{f}=(f_1,\cdots,f_m)$ be a collection of m functions on $\mathbb{R}^n$. Given $\alpha\in(0,mn)$ and $(b_1,\cdots,b_m)\in(BMO)^m$. We consider the following multilinear fractional integral operators $I_{\alpha,m}$ defined by

$$I_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}dy_1\cdots dy_m. \tag{1.1}$$

The corresponding iterated commutators $I^{\Pi b}_{\alpha,m}$ defined by

$$I^{\Pi b}_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{\prod_{i=1}^m(b_i(x)-b_i(y_i)f_i(y_i)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}dy_1\cdots dy_m. \tag{1.2}$$

As is well known, multilinear fractional integral operator was first studied by Grafakos [1], subsequently, by Kenig and Stein [2], Grafakos and Kalton [3]. In 2009, Moen [4] introduced weight function $A_{\vec{p},q}$ and gave weighted inequalities for multilinear

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fractional integral operators; In 2013, Chen and Wu [5] obtained the weighted norm inequalities for the iterated commutators $I^{\Pi b}_{\alpha,m}$. More results of the weighted inequalities for multilinear fractional integral and commutators can be found in [6–9].

We list some results mentioned above.

Theorem 1.1 (see [4]). *Let $m\geq 2$ and $0<\alpha<mn$. Suppose $1/p=1/p_1+\cdots+1/p_m$, $1/q=1/p-\alpha/n$, $\vec{\omega}=(\omega_1,\cdots,\omega_m)$ satisfies the $A_{\vec{p},q}$ condition, and $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i$. If $p_1,\cdots,p_m\in(1,\infty)$, then there exists a constant C independent of $\vec{f}=(f_1,\cdots,f_m)$ such that*

$$\|I_{\alpha,m}\vec{f}\|_{L^q(\nu^q_{\vec{\omega}})}\leq C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \tag{1.3}$$

Theorem 1.2 (see [5]). *Let $0<\alpha<mn$ and $(b_1,\cdots,b_m)\in(BMO)^m$. For $1<p_1,\cdots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$, and $1/q=1/p-\alpha/n$, if $\vec{\omega}\in A_{\vec{p},q}$, then there exists a constant $C>0$ such that*

$$\|I^{\Pi b}_{\alpha,m}(\vec{f})\|_{L^q(\nu^q_{\vec{\omega}})}\leq C\prod_{i=1}^m\|b_i\|_*\|f_i\|_{L^{p_i}(\omega_i^{p_i})}, \tag{1.4}$$

where $\nu_{\vec{\omega}}=\prod_{i=1}^m\omega_i$.

Let $B_k=\{x\in\mathbb{R}^n:\ |x|\leq 2^k\}$ and $C_k=B_k\setminus B_{k-1}$ for any $k\in\mathbb{Z}$. Denote $\chi_k=\chi_{C_k}$ for $k\in\mathbb{Z}$, where χ_{C_k} is the characteristic function of the set C_k. The following weighted Herz space was introduced by Lu and Yang in [10].

Let $\sigma\in\mathbb{R}$, $0<p,q<\infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\sigma,p}_q(\omega_1,\omega_2)$ is defined by

$$\dot{K}^{\sigma,p}_q(\omega_1,\omega_2)=\{f\in L^q_{loc}(\mathbb{R}^n\setminus\{0\},\omega_2):\ \|f\|_{\dot{K}^{\sigma,p}_q(\omega_1,\omega_2)}<\infty\},$$

where

$$\|f\|_{\dot{K}^{\sigma,p}_q(\omega_1,\omega_2)}=\Big(\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\sigma p/n}\|f\chi_k\|^p_{L^q_{\omega_2}}\Big)^{1/p}.$$

In 2000, Lu, Yabuta and Yang in [11] obtained boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators and their commutators on weighted Herz type spaces. Wang in [12] proved that the intrinsic square functions are bounded on weighted Herz type Hardy spaces. In [13], Hu and Wang considered parametric Marcinkiewicz integral and its commutator on Weighted Herz spaces.

For a sequence $\varphi=\{\varphi(k)\}^{\infty}_{-\infty}$, $\varphi(k)>0$. We suppose that φ satisfies doubling condition of order (a,b) and write $\varphi\in D(a,b)$ if there exists $C\geq 1$ such that

$$C^{-1}2^{a(k-j)}\leq\frac{\varphi(k)}{\varphi(j)}\leq C2^{b(k-j)} \tag{1.5}$$