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Commutators of Lipschitz Functions and Singular Integrals with Non-Smooth Kernels on Euclidean Spaces

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Abstract. In this article, we obtain the L^p-boundedness of commutators of Lipschitz functions and singular integrals with non-smooth kernels on Euclidean spaces.

Key Words: Commutators, singular integrals, maximal functions, sharp maximal functions, muckenhoupt weights, Lipschitz spaces.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

Consider the singular integral operator T defined by

$$Tf(x)=\int_{\mathbb{R}^n} K(x,y)f(y)dy, \tag{1.1}$$

where f is a continuous function with compact support, $x\notin \operatorname{supp} f$; and the kernel $K(x,y)$ is a measurable function defined on $(\mathbb{R}^n\times\mathbb{R}^n)\backslash\Delta$ with $\Delta=\{(x,x):x\in\mathbb{R}^n\}$. If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b,T]$ of a BMO function b and the singular integral operator T is defined by

$$T_b f:=[b,T](f):=T(bf)-bT(f).$$

The L^p-boundedness $(1<p<\infty)$ of T and T_b are well known in the Euclidean setting, provided that the kernel $K(x,y)$ of the operator T satisfies Hörmander's conditions (see [1, 15–17] among many other good references). In 1999, Duong and McIntosh [3] obtained the L^p-boundedness of T, under the assumption that the kernel $K(x,y)$ satisfies some conditions which are weaker than Hörmander's integral conditions. The boundedness of the operator T with non-smooth kernel on $L^p(w)$ $(w\in\mathcal{A}_p(\mathbb{R}^n)$, $1<p<\infty)$ was proved by Martell [12]. Moreover, Duong and Yan [4] obtained the L^p-boundedness of

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the commutator T_b under some conditions which are weaker than Hörmander's pointwise conditions. Lin and Jiang [11] also obtained the L^p-boundedness of T_b, but with $b\in \mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)$. See also [8,9,13,18] for additional results on these topics.

The purpose of this paper is to extend the results in [11]. That is, we would like to obtain the L^p-boundedness $(1<p<\infty)$ of the operator $T_{\vec{b}}$, where

$$T_{\vec{b}}f(x)=\int_{\mathbb{R}^n}\Big\{\prod_{i=1}^{k}(b_i(x)-b_i(y))\Big\}K(x,y)f(y)dy, \tag{1.2}$$

$b_i\in \mathrm{Lip}_{\alpha_i,w}(\mathbb{R}^n)$ for $1\le i\le k$, and the weight w belongs to a subclass of $\mathcal{A}_1$.

2 Background

2.1 $\mathcal{A}_p$ weights

For a ball B in $\mathbb{R}^n$, let $|B|$ denote the measure of the ball B. A weight w is said to belong to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^n)$, $1<p<\infty$, if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w^{-p'/p}(x)dx\right)^{p/p'}\le C<\infty,$$

for all balls B in $\mathbb{R}^n$. The smallest constant C for which the above inequality holds is the $\mathcal{A}_p$ bound of w. The class $\mathcal{A}_1(\mathbb{R}^n)$ consists of non-negative functions w such that

$$\frac{w(B)}{|B|}:=\frac{1}{|B|}\int_B w(x)dx\le C\ \text{ess}\inf_{x\in B}w(x)$$

for all balls B in $\mathbb{R}^n$. It is well-known that (see [7,17] for instance) if $w\in\mathcal{A}_p(\mathbb{R}^n)$ for some $p\in[1,\infty)$, then for any measurable subset $E\subset B$, there exist positive constants γ and C such that

$$\frac{w(E)}{w(B)}\le C\left(\frac{|E|}{|B|}\right)^{\gamma}. \tag{2.1}$$

Inequality (2.1) indeed holds with $\gamma\in(0,1)$. This will be used in the estimate of (3.3) below. Furthermore, if $w\in\mathcal{A}_p(\mathbb{R}^n)$ $(1\le p\le\infty)$, then it satisfies the reverse Hölder inequality. That is, there exist $s'>1$ and $c>0$ (both depending on w) so that

$$\left(\frac{1}{|B|}\int_B w(x)^{s'}dx\right)^{1/s'}\le\frac{c}{|B|}\int_B w(x)dx\quad\text{for all balls }\ B\subset\mathbb{R}^n. \tag{2.2}$$

A weight w is said to belong to the class $\mathcal{A}_{p,q}(\mathbb{R}^n)$, $1<p,q<\infty$, if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B w^q(x)dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w^{-p'}(x)dx\right)^{1/p'}\le C<\infty,$$