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On Growth of Polynomials with Restricted Zeros

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z|<k$, $k\geq 1$. It is known that for each $0\leq s<n$ and $1\leq R\leq k$,

$$M(P^{(s)},R)\leq\Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{1+k}\Big)^n M(P,1).$$

In this paper, we obtain certain extensions and refinements of this inequality by involving binomial coefficients and some of the coefficients of the polynomial $P(z)$.

Key Words: Polynomial, maximum modulus princple, zeros.

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1 Introduction and statement of results

Let P_n be the class of polynomials

$$P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$$

of degree n, z being a complex variable and $P^{(s)}(z)$ be its s^{th} derivative. For $P\in P_n$, let $M(P,R)=\max_{|z|=R}|P(z)|$. It is well known that

$$M(P',1)\leq nM(P,1), \tag{1.1}$$

and

$$M(P,R)\leq R^n M(P,1),\quad R\geq 1. \tag{1.2}$$

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The inequality (1.1) is a famous result of S. Bernstein (for reference, see [9]) whereas the inequality (1.2) is a simple consequence of Maximum Modulus Principle (see [8]). It was shown by Ankeny and Rivlin [1] that if $P\in P_n$ and $P(z)\neq 0$ in $|z|<1$, then (1.2) can be replaced by

$$M(P,R)\leq\Big(\frac{R^n+1}{2}\Big)(P,1),\quad R\geq 1. \tag{1.3}$$

Recently, Jain [5] obtained a generalization of (1.3) by considering polynomials with no zeros in $|z|<k$, $k\geq 1$ and simultaneously have taken into consideration the s^{th} derivative of the polynomial, $(0\leq s<n)$, instead of the polynomial itself. More precisely, he proved the following result.

Theorem 1.1. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then for $0\leq s<n$,*

$$M\big(P^{(s)},R\big)\leq\frac{1}{2}\Big\{\frac{d^{(s)}}{dR^{(s)}}(R^n+k^n)\Big\}\Big(\frac{2}{1+k}\Big)^n M(P,1)\quad \textit{for}\ \ R\geq k, \tag{1.4}$$

and

$$M\big(P^{(s)},R\big)\leq\Big(\frac{1}{R^s+k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{1+k}\Big)^n M(P,1)\quad \textit{for}\ \ 1\leq R\leq k. \tag{1.5}$$

Equality holds in (1.4) *(with $k=1$ and $s=0$) for $P(z)=z^n+1$ and equality holds in* (1.5) *(with $s=1$) for $P(z)=(z+k)^n$.*

In this paper, we obtain certain extensions and refinements of the inequality (1.5) of the above theorem by involving binomial coefficients and some of the coefficients of polynomial $P(z)$. More precisely, we prove

Theorem 1.2. *If $P\in P_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\leq s<n$ and $0<r\leq R\leq k$, we have*

$$\begin{aligned}M\big(P^{(s)},R\big)\leq&\Big\{\frac{c(n,s)R+\big|\frac{a_s}{a_0}\big|k^{s+1}}{c(n,s)\big(k^{s+1}+R^{s+1}\big)+\big|\frac{a_s}{a_0}\big|\big(k^{s+1}R^s+Rk^{2s}\big)}\Big\}\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\\&\times\Big\{\exp\Big(n\int_r^R\frac{t+\frac{1}{n}\big|\frac{a_1}{a_0}\big|k^2}{t^2+k^2+\frac{2k^2}{n}\big|\frac{a_1}{a_0}\big|t}dt\Big)\Big\}M(P,r).\end{aligned} \tag{1.6}$$

The result is best possible (with $s=1$) and equality in (1.6) *holds for $P(z)=(z+k)^n$.*

Remark 1.1. Since if $P(z)\neq 0$ in $|z|<k$, $k>0$, then by Lemma 2.5 (stated in Section 2), we have for $0\leq s<n$,

$$\frac{1}{c(n,s)}\Big|\frac{a_s}{a_0}\Big|k^s\leq 1, \tag{1.7}$$