Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 3 (2016), pp. 272-282
DOI: 10.4208/ata.2016.v32.n3.5

An Estimate on Riemannian Manifolds of Dimension 4

Samy Skander Bahoura*

Departement de Mathematiques, Universite Pierre et Marie Curie, 2 place Jussieu, 75005, Paris, France

Received 31 July 2015; Accepted (in revised version) 28 July 2016

Abstract. We give an estimate of type $\sup\times\inf$ on Riemannian manifold of dimension 4 for a Yamabe type equation.

Key Words: $\sup\times\inf$, Riemannian manifold, dimension 4.

AMS Subject Classifications: 53C21, 35J60 35B45 35B50

1 Introduction and main results

In this paper, we deal with the following Yamabe type equation in dimension $n=4$:

$$\Delta_g u+hu=8u^3,\quad u>0. \tag{1.1}$$

Here, $\Delta_g=-\nabla^i(\nabla_i)$ is the Laplace-Beltrami operator and h is an arbitrary bounded function.

The Eq. (1.1) was studied a lot, when $M=\Omega\subset\mathbb{R}^n$ or $M=\mathbb{S}_n$ see for example, [2–4, 11, 15]. In this case we have a $\sup\times\inf$ inequality. The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u=V(x)e^u. \tag{1.2}$$

The Eq. (1.2) was studied by many authors and we can find very important result about a priori estimates in [8,9,12,16] and [19]. In particular in [9], we have the following interior estimate:

$$\sup_K u\le c=c\Big(\inf_\Omega V,\|V\|_{L^\infty(\Omega)},\inf_\Omega u,K,\Omega\Big).$$

And, precisely, in [8,12,16] and [20], we have:

$$C\sup_K u+\inf_\Omega u\le c=c\Big(\inf_\Omega V,\|V\|_{L^\infty(\Omega)},K,\Omega\Big),$$

*Corresponding author. *Email address:* samybahoura@yahoo.fr (S. S. Bahoura)

and

$$\sup_K u + \inf_\Omega u \le c = c\Big(\inf_\Omega V, \|V\|_{C^\alpha(\Omega)}, K, \Omega\Big),$$

where K is a compact subset of Ω, C is a positive constant which depends on

$$\frac{\inf_\Omega V}{\sup_\Omega V}$$

and $\alpha \in (0,1]$. When $6h = R_g$ the scalar curvature, and M compact, the Eq. (1.1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness results for Eq. (1.1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M a Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup \times \inf$ is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3,6,10,11,16,17], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type $\sup \times \inf$ for the Eq. (1.1) in dimension 4. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3,5]. There are other estimates of type $\sup + \inf$ on complex Monge-Ampere equation on compact manifolds, see [20,21]. They consider, on compact Kahler manifold (M,g), the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f - t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \quad \text{on } M. \end{cases} \tag{1.3}$$

And, they prove some estimates of type $\sup_M + m\inf_M \le C$ or $\sup_M + m\inf_M \ge C$ under the positivity of the first Chern class of M. Here, we have,

Theorem 1.1. *For all compact set K of M, there is a positive constant c, which depends only on, $h_0 = \|h\|_{L^\infty(M)}$, K, M, g such that:*

$$\Big(\sup_K u\Big)^{1/3} \times \inf_M u \le c,$$

for all u solution of (1.1).

Here we consider more general equation and this theorem extends a result of Li and Zhang, see [17]. Li and Zhang considered precisely the Yamabe equation and here we consider a general equation ($h \neq \frac{1}{6} R_g$ with R_g the scalar curvature). Here, we use a different method than the method of Li and Zhang in [17]. Also, we extend a result of [5].