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Characterizations of Function Spaces via Boundedness of Commutators

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Abstract. In this paper, we give some creative characterizations of Campanato spaces via the boundedness of commutators associated with the Calderón-Zygmund singular integral operator, fractional integrals and Hardy type operators. Furthermore, we put forward a few problems on the characterizations of Campanato type spaces via the boundedness of commutators.

Key Words: Campanato space, Morrey space, commutator.

AMS Subject Classifications: 42B20, 42B25

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1 Introduction

Let $-n/p\leq\beta<1$ and $1\leq p<\infty$. Then the Campanato space $C^{p,\beta}(\mathbb{R}^n)$ was defined by the norm

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)}=\sup_B\|f\|_{C^{p,\beta}(B)}:=\sup_B\frac{1}{|B|^{\frac{\beta}{n}}}\left(\frac{1}{|B|}\int_B|f-f_B|^p dx\right)^{1/p}, \tag{1.1}$$

where

$$f_B=\frac{1}{|B|}\int_B f(x)dx,$$

B denotes any ball contained in $\mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. Campanato spaces are useful tools in the regularity theory of PDE as a result of their better structures that allow to give an integral characterization of the spaces of Hölder continuous functions, which leads to a generalization of the classical Sobolev embedding theorem, see e.g., [20] and Lu's work (see [23, 24]). It is also well known that $C^{1,1/p-1}$ is the dual

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space of Hardy space $H^p(\mathbb{R}^n)$ $(0<p<1)$ (see [36]). For a recent account of the theory on $C^{p,\beta}(\mathbb{R}^n)$, we refer the reader to [11,21,28,39] and the references therein.

Many important functional spaces are special cases of Campanato space. In fact, we have

$$C^{p,\beta}(\mathbb{R}^n)=\begin{cases} BMO(\mathbb{R}^n), & \beta=0,\\ Lip_\beta(\mathbb{R}^n), & 0<\beta<1,\\ \supset M^{p,\beta}(\mathbb{R}^n), & -n/p\le\beta<0.\end{cases}$$

Here $BMO(\mathbb{R}^n)$ denote the space of Bounded Mean Oscillation functions, $Lip_\beta(\mathbb{R}^n)$, $0<\beta<1$, is the Lipschitz functional space and $M^{p,\beta}(\mathbb{R}^n)$, $-n/p\le\beta<0$, is the Morrey space with the following norm

$$\|f\|_{M^{p,\beta}(\mathbb{R}^n)}=\sup_B\frac{1}{|B|^{\frac{\beta}{n}}}\left(\frac{1}{|B|}\int_B|f(x)|^pdx\right)^{1/p}.$$

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be an integral operator. Then the commutator operator formed by T and b was denoted by

$$[b,T](f):=bTf-T(bf).$$

The function b was also called the symbol function of $[b,T]$. The investigation of the operator $[b,T]$ begin with Calderón-Zygmund pioneering study of the operator T (see [4] and [8]). They found that the theory of commutators play an important role in studying the regularity of solutions to elliptic PDEs of the second order. The well-posedness of solutions to many PDEs can be attributed to the corresponding commutator's boundedness for singular integral operators. However, this topic exceeds the scope of this paper, for more information about this, see for example [3, 10, 12] and [35]. Especially in [35], the authors simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly by some estimates of commutators which were obtained by Yan in his Ph.D. thesis [38] (see also Lu and Yan's work in [27]). Since $L^\infty(\mathbb{R}^n)\subsetneq BMO(\mathbb{R}^n)$, the boundedness of $[b,T]$ is worse than T (for example, the singularity, see also [30]). Therefore, many authors want to know whether $[b,T]$ shares the similar boundedness with T. Many authors are interested in the study of commutators when the symbol functions b belong to BMO spaces and Lipschitz spaces. For some of this classical works, we refer the reader to [1,18,25] and [29].

In this paper, we focus on some characterizations of Campanato spaces via the boundedness of $[b,T]$ when

T is Calderón-Zygmund singular integral operator;

T is fractional integral;

T is Hardy type operator.

Throughout this paper, for $x_0\in\mathbb{R}^n$, $r>0$ and $\lambda>0$, $B=B(x_0,r)$ denotes the ball centered at x_0 with radius r and $\lambda B=B(x_0,\lambda r)$. C is a constant which may change from line to line.