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The Multifractal Formalism for Measures, Review and Extension to Mixed Cases

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Abstract. The multifractal formalism for single measure is reviewed. Next, a mixed generalized multifractal formalism is introduced which extends the multifractal formalism of a single measure based on generalizations of the Hausdorff and packing measures to a vector of simultaneously many measures. Borel-Cantelli and Large deviations Theorems are extended to higher orders and thus applied for the validity of the new variant of the multifractal formalism for some special cases of multi-doubling type measures.

Key Words: Hausdorff measures, packing measures, Hausdorff dimension, packing dimension, renyi dimension, multifractal formalism, vector valued measures, mixed cases, Holderian measures, doubling measures, Borel-Cantelli, large deviations.

AMS Subject Classifications: 28A78, 28A80

1 Introduction

In the present work, we are concerned with the whole topic of multifractal analysis of measures and the validity of multifractal formalisms. We aim to consider some cases of simultaneous behaviors of measures instead of a single measure as in the classic or original multifractal analysis of measures. We call such a study mixed multifractal analysis. Such a mixed analysis has been generating a great attention recently and thus

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proved to be powerful in describing the local behavior of measures especially fractal ones (see [1, 2, 9–14]).

In this paper, multi purposes will be done. Firstly we review the classical multifractal analysis of measures and recall all basics about fractal measures as well as fractal dimensions. We review Hausdorff measures, Packing measures, Hausdorff dimensions, Packing dimensions as well as Renyi dimensions and we recall the eventual relations linking these notions. A second aim is to develop a type of multifractal analysis, multifractal spectra, multifractal formalism which permit to study simultaneously a higher number of measures. As it is noticed from the literature on multifractal analysis of measures, this latter always considered a single measure and studies its scaling behavior as well as the multifractal formalism associated. Recently, many works have been focused on the study of simultaneous behaviors of finitely many measures. In [9], a mixed multifractal analysis is developed dealing with a generalization of Rényi dimensions for finitely many self similar measures. This was one of the motivations leading to our present paper. Secondly, we intend to combine the generalized Hausdorff and packing measures and dimensions recalled after with Olsen's results in [14] to define and develop a more general multifractal analysis for finitely many measures by studying their simultaneous regularity, spectrum and to define a mixed multifractal formalism which may describe better the geometry of the singularities's sets of these measures. We apply the techniques of L. Olsen especially in [9] and [14] with the necessary modifications to give a detailed study of computing general mixed multifractal dimensions of simultaneously many finite number of measures and try to project our results for the case of a single measure to show the generecity of our's.

The first point to check in multifractal analysis of a measure is its singularity on its spectrum. Given a measure μ eventually Borel and finite, for $x\in\operatorname{supp}(\mu)$, the singularity of μ is estimated via $\mu(B(x,r))$ as $r\to 0$. If $\mu(B(x,r))\sim r^{\alpha}$, the measure μ is said to be α-Hölder at x. The local lower dimension and the local upper dimension of μ at the point x are respectively defined by

$$\underline{\alpha}_{\mu}(x)=\liminf_{r\downarrow 0}\frac{\log(\mu(B(x,r)))}{\log r} \quad \text{and} \quad \overline{\alpha}_{\mu}(x)=\limsup_{r\downarrow 0}\frac{\log(\mu(B(x,r)))}{\log r}.$$

When these quantities are equal we call their common value the local dimension, denoted by $\alpha_{\mu}(x)$ of μ at x. Next, the α-singularity set is $X(\alpha)=\{x\in\operatorname{supp}(\mu);\ \alpha_{\mu}(x)=\alpha\}$ and finally, the spectrum of singularities is the mapping defined by $d(\alpha)=\dim X(\alpha)$ where dim stands for the Hausdorff dimension.

The computation of such a spectrum is the delicate point and the most principal aim in the whole multifractal study of the measure. Its computation needs more efforts and special techniques based on the characteristics of the measure, such that self similarity, scalings. In multifractal analysis, it is related to multifractal dimensions and in some cases it is computed by means of the Legendre transform of such dimensions. This fact constitutes the so-called multifractal formalism for measures.