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A Perturbation of Jensen $*$-Derivations from $K(H)$ into $K(H)$

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Abstract. Let's take H as an infinite–dimensional Hilbert space and $K(H)$ be the set of all compact operators on H. Using Spectral theorem for compact self–adjoint operators, we prove the Hyers–Ulam stability of Jensen $*$-derivations from $K(H)$ into $K(H)$.

Key Words: Jensen $*$-derivation, C^*-algebra, Hyers–Ulam stability.
AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In a Hilbert space H, an operator T in $B(H)$ is called a compact operator if the image of unit ball of H under T is a compact subset of H. Note that if the operator $T:H\longrightarrow H$ is compact, then the adjoint of T is compact, too. The set of all compact operators on H is shown by $K(H)$. It is easy to see that $K(H)$ is a C^*-algebra [1]. Moreover, every operator on H with finite range is compact. The set of all finite range projections on Hilbert space H is denoted by $P(H)$.

An approximate unit for a C^*-algebra $\mathcal{A}$ is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $a=\lim_\lambda au_\lambda=\lim_\lambda u_\lambda a$ for all $a\in\mathcal{A}$. Every C^*-algebra admits an approximate unit [2].

Example 1.1. Let H be a Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. The C^*-algebra $K(H)$ is non–unital since $dim(H)=\infty$. If P_n is a projection on $\mathbb{C}e_1+\cdots+\mathbb{C}e_n$, then the increasing sequence $(P_n)_{n=1}^\infty$ is an approximate unit for $K(H)$.

Theorem 1.1 (see [2]). *Let $T:H\longrightarrow H$ be a compact self–adjoint operator on Hilbert space H. Then there is an orthonormal basis of H consisting of eigenvectors of T. The nonzero eigenvalues of T are from finite or countably infinite set $\{\lambda_k\}_{k=1}^\infty$ of real numbers and $T=\sum_{k=1}^\infty\lambda_kP_k$, where P_k is the orthogonal projection on the finite–dimensional space of eigenvectors corresponding to eigenvalues. If the number of nonzero eigenvalues is countably infinite, then the series converges to T in the operator norm.*

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The problem of stability of functional equations originated from a question of Ulam [5] concerning the stability of group homomorphisms: let $(G1,*)$ be a group and let $(G2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta(\varepsilon)>0$ such that if a mapping $h:G_1\longrightarrow G_2$ satisfies the inequality

$$d(h(x*y),h(x)\star h(y))<\delta$$

for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\varepsilon$$

for all $x\in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y)=H(x)\star H(y)$ is stable. Thus, the stability question of functional equations is that how the solutions of the inequality differ from those of the given functional equation.

Hyers [3] gave the first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f:X\longrightarrow Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x,y\in X$ and some $\varepsilon>0$. Then, there exists a unique additive mapping $T:X\longrightarrow Y$ such that

$$\|f(x)-T(x)\|\le\varepsilon$$

for all $x\in X$. Also, if the function $t\longmapsto f(tx)$ from $\mathbb{R}$ to Y is continuous for each fixed $x\in X$, then T is an $\mathbb{R}$-linear function. This method is called the direct method or Hyers–Ulam stability of functional equations.

Note that if f is continuous, then the function $r\mapsto f(rx)$ from $\mathbb{R}$ into Y is continuous for all $x\in X$. Therefore T is $\mathbb{R}$-linear.

Definition 1.1. Let X and Y be real linear spaces. For $n\in\{2,3,4,\cdots\}$ the mapping $f:X\longrightarrow Y$ is called a Jensen mapping of n-variable, if f for each $x_1,\cdots,x_n\in X$ satisfies the following equation

$$f\Big(\frac{x_1+\cdots+x_n}{n}\Big)=\frac{1}{n}\big(f(x_1)+\cdots+f(x_n)\big).$$

In 2003, J. M. Rassias and M. J. Rassias [4] investigated the Ulam stability of Jensen and Jensen type mappings by applying the Hyers method. In 2012, M. Eshaghi Gordji and S. Abbaszadeh [6] investigated the Hyers–Ulam stability of Jensen type and generalized n-variable Jensen type functional equations in fuzzy Banach spaces.

Definition 1.2. Let $\mathcal{A}$ be a C^*-algebra. A mapping $d:\mathcal{A}\longrightarrow\mathcal{A}$ with $d(a^*)=d(a)^*$ for all $a\in\mathcal{A}$ ($*$-preserving property) is called a Jensen $*$-derivation if d satisfies

$$d(x_1x_2)=x_1d(x_2)+d(x_1)x_2$$