Classical Fourier Analysis over Homogeneous Spaces of Compact Groups

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Abstract. This paper introduces a unified operator theory approach to the abstract Fourier analysis over homogeneous spaces of compact groups. Let $G$ be a compact group and $H$ be a closed subgroup of $G$. Let $G/H$ be the left coset space of $H$ in $G$ and $\mu$ be the normalized $G$-invariant measure on $G/H$ associated to the Weil’s formula. Then, we present a generalized abstract framework of Fourier analysis for the Hilbert function space $L^2(G/H,\mu)$.

Key Words: Compact group, homogeneous space, dual space, Fourier transform, Plancherel (trace) formula, Peter-Weyl Theorem.

AMS Subject Classifications: 20G05, 43A85, 43A32, 43A40, 43A90

1 Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or precisely left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis [2–4, 12], theoretical and particle physics [1, 9–11, 13]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics and scientific computing (computational engineering), see [6, 7, 13–18] and references therein.

Let $G$ be a compact group, $H$ be a closed subgroup of $G$, and $\mu$ be the normalized $G$-invariant measure on $G/H$ associated to the Weil’s formula. The left coset space $G/H$ is considered as a compact homogeneous space, which $G$ acts on it via the left action. This paper which contains 5 sections, is organized as follows. Section 2 is devoted to fix notations and preliminaries including a brief summary on Hilbert-Schmidt operators,

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non-Abelian Fourier analysis over compact groups, and classical results on abstract harmonic analysis over locally compact homogeneous spaces. We present some abstract harmonic analysis aspects of the Hilbert function space $L^2(G/H, \mu)$, in Section 3. Then we define the abstract notion of dual space $\hat{G}/H$ for the homogeneous space $G/H$ and we will show that this definition is precisely the standard dual space for the compact quotient group $G/H$, when $H$ is a closed normal subgroup of $G$. We then introduce the definition of abstract operator-valued Fourier transform over the Banach function space $L^1(G/H, \mu)$ and also generalized version of the abstract Plancherel (trace) formula for the Hilbert function space $L^2(G/H, \mu)$. The paper closes by a presentation of Peter-Weyl Theorem for the Hilbert function space $L^2(G/H, \mu)$.

2 Preliminaries and notations

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k \|Te_k\|^2 < \infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $HS(\mathcal{H})$ and for $T \in HS(\mathcal{H})$ the Hilbert-Schmidt norm of $T$ is $\|T\|_{HS}^2 = \sum_k \|Te_k\|^2$. The set $HS(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$ and if $\mathcal{H}$ is finite-dimensional we have $HS(\mathcal{H}) = \mathcal{B}(\mathcal{H})$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is trace-class, whenever $\|T\|_t = \text{tr}\|T\| < \infty$, if $\text{tr}[T] = \sum_k \langle Te_k, e_k \rangle$ and $|T| = (TT^*)^{1/2}$ [20].

Let $G$ be a compact group with the probability Haar measure $dx$. Then each irreducible representation of $G$ is finite dimensional and every unitary representation of $G$ is a direct sum of irreducible representations, see [1,10]. The set of of all unitary equivalence classes of irreducible unitary representations of $G$ is denoted by $\hat{G}$. This definition of $\hat{G}$ is in essential agreement with the classical definition when $G$ is Abelian, since each character of an Abelian group is a one dimensional representation of $G$. If $\pi$ is any unitary representation of $G$, for $\zeta, \xi \in \mathcal{H}_\pi$ the functions $\pi_{\zeta, \xi}(x) = \langle \pi(x)\zeta, \xi \rangle$ are called matrix elements of $\pi$. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then $\pi_{ij}$ means $\pi_{e_i, e_j}$. The notation $E_\pi$ is used for the linear span of the matrix elements of $\pi$ and the notation $E$ is used for the linear span of $\cup_{[\pi] \in \hat{G}} E_\pi$. Then Peter-Weyl Theorem [1,10] guarantees that if $G$ is a compact group, $E$ is uniformly dense in $E(G)$, $L^2(G) = \bigoplus_{[\pi] \in \hat{G}} E_\pi$, and $\{d_\pi^{-1/2} \pi_{ij} : i,j = 1, \ldots, d_\pi, [\pi] \in \hat{G}\}$ is an orthonormal basis for $L^2(G)$. For $f \in L^1(G)$ and $[\pi] \in \hat{G}$, the Fourier transform of $f$ at $\pi$ is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\hat{f}(\pi) = \int_G f(x) \pi(x)^* dx. \quad (2.1)$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x)) \in \mathbb{C}^{d_\pi \times d_\pi}$. Then $\hat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is the matrix with entries given by $\hat{f}(\pi)_{ij} = d_\pi^{-1} c_{ji}^\pi(f)$ which satisfies

$$\sum_{i,j=1}^{d_\pi} c_{ij}^\pi(f) \pi_{ij}(x) = d_\pi \sum_{i,j=1}^{d_\pi} \hat{f}(\pi)_{ij} \pi_{ij}(x) = d_\pi \text{tr}[\hat{f}(\pi) \pi(x)],$$