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Weighted Best Local Approximation

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Abstract. In this survey the notion of a balanced best multipoint local approximation is fully exposed since they were treated in the L^p spaces and recent results in Orlicz spaces. The notion of balanced point, introduced by Chui et al. in 1984 are extensively used.

Key Words: Best Local approximation, multipoint approximation, balanced neighborhood.

AMS Subject Classifications: 41A10, 41A65

1 Introduction

The notion of a best multipoint local approximations of a function is fully treated in [2] where the L^p norm is used. Later, other approaches to best multipoint local approximations with L^p norms appeared in [7] and [8]. And finally, for Orlicz norms, we mention [3,5,9,12,13] and for a general family of norms [6,10] and [14]. However, in [2], Chui et al. introduced the concept of balanced points in L^p which includes different importance in each point.

More precisely, a rather general view of the problem is as follows. Let $f:\mathbb{R}\to\mathbb{R}$ be a function in a normed space X with norm $\|\cdot\|$. Let Π^m denote the set of polynomial in $\mathbb{R}$ of degree less or equal than m and suppose $\Pi^m\subseteq X$. Consider n points $x_1,\cdots,x_n$ in $\mathbb{R}$ and a net of small Lebesgue measurable neighborhoods V_i^δ around each point x_i such that the Lebesgue measure $|V_i^\delta|$ goes to 0 as $\delta\to 0$ for $i=1,\cdots,n$. We select the best approximation to f near the points $x_1,\cdots,x_n$ by polynomial in Π^m. Formally, for each n-tuple of neighborhoods $V_1,\cdots,V_n$ we consider the polynomial $g_V\in\Pi^m$ which minimizes

$$\|(f-h)\mathcal{X}_V\| \tag{1.1}$$

for all $h\in\Pi^m$, where $V=\cup_{i=1}^n V_i$ and $V_i=V_i^\delta$. It is well known that a best $\|\cdot\|$-approximation g_V always exists since Π^m has finite dimension. If any net g_V converges to a unique

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element $g\in\Pi^m$, as $\delta\to 0$, then g is said to be a best local approximation to f at the points $x_1,\cdots,x_n$. We will mention in this survey all the works which consider that the velocity of convergence $|V_i|\to 0$, as $|V|\to 0$, can be different at each point x_i. According to [2], this problem has been treated considering the concept of balanced neighborhoods in local approximation and it reflects the different importance of the points $x_1,\cdots,x_n$. We need to deal with the necessary definition of balanced neighborhoods in each context.

As we pointed out above, Chui et al. study in [2] this problem when the space X is the usual L^p space, with the norm $\|f\|_p=\left(\int_B|f(x)|^p dx\right)^{1/p}$, where B is a measurable set. They get results for balanced and non balanced neighborhoods. At last they generalize the results to the case of $\mathbb{R}^k$ instead of $\mathbb{R}$. On the other hand, in [4], the authors get balanced results in L^p using other technique. We will discuss the L^p problem with more details in Section 3.

In [11] and [12] the authors study the problem in Orlicz spaces, it means,

$$X=L^{\phi}(B):=\Big\{f:\ \int_B\phi(\alpha|f(x)|)dx<\infty\ \text{ for some }\ \alpha>0\Big\},$$

where ϕ is a convex function, non negative, defined on $\mathbb{R}_0^+$ and B is a Lebesgue measurable set. In these two works, the authors studied the best local approximation problem with the Luxemburg norm

$$\|f\|_\phi=\|f\|_{L^\phi(B)}=\inf\Big\{\lambda>0:\ \int_B\phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\Big\} \tag{1.2}$$

and get results for balanced and non balanced neighborhoods. Furthermore, we can consider a different Luxemburg norms in Orlicz Spaces, that is

$$\|f\|_{\phi,B}=\inf\Big\{\lambda>0:\ \int_B\phi\left(\frac{|f(x)|}{\lambda}\right)dx\le |B|\Big\} \tag{1.3}$$

and it can generate different best approximation functions g_V than those obtained with the standard Luxemburg norm given in (1.2). In [13] the authors study the balanced neighborhoods problem with this norm using a different technique than that used in [11] and [12].

Moreover, in [9] the authors study the balanced problem in Orlicz spaces L^ϕ when the error (1.1) does not come from a norm, but considering

$$\int_V\phi(|f(x)-g_V(x)|)\,dx=\min_{h\in\pi^n}\int_V\phi(|f(x)-h(x)|)\,dx.$$

The last three problems in L^p are equivalent, but in Orlicz spaces they are different problems and have different concepts of balanced neighborhoods. In Section 4 we will present the three problems in Orlicz spaces in detail.