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Algorithms and Identities for (q,h)-Bernstein Polynomials and (q,h)-Bézier Curves–A Non-Blossoming Approach

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Abstract. We establish several fundamental identities, including recurrence relations, degree elevation formulas, partition of unity and Marsden identity, for quantum Bernstein bases and quantum Bézier curves. We also develop two term recurrence relations for quantum Bernstein bases and recursive evaluation algorithms for quantum Bézier curves. Our proofs use standard mathematical induction and other elementary techniques.

Key Words: Bernstein polynomials, Bézier curves, Marsden's identity, recursive evaluation.

AMS Subject Classifications: 11C08, 65DXX, 65D15, 65D17

1 Introduction and definitions

Bernstein bases are polynomial bases used as blending functions for the construction of Bézier curves and surfaces. These bases have been used extensively over the last half century in geometric modeling, computer aided geometric design (CAGD), and approximation theory. The main application of Bézier curves and surfaces is in mathematical modeling of curves and surfaces that are used in various real life problems. One essential property of a Bézier curve or a Bézier surface is that it can be computed very efficiently using affine recursive evaluation algorithms. This is due to certain structural properties of the Bernstein basis functions that other polynomial bases do not possess.

The classical Bernstein polynomials were introduced by Bernstein in 1912 and have found many applications in applied and computational mathematics since then. The classical Bézier curves and surfaces were introduced in 1962 by the French engineer Pierre

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Bézier who worked for the French car manufacturer Renault. He used Bézier curves and surfaces to design and model aerodynamic car bodies [1]. The q-Bernstein polynomials were introduced and studied only recently by G. Phillips and his collaborators [7]. The theory of quantum q- and h-Bézier curves in the context of the quantum q- and h-blossoming was developed very recently by Goldman, Simeonov, and Zafiris [4,9,10].

In this paper, our main goal is to state and prove several of the most important properties of the (q,h)-Bernstein polynomials and (q,h)-Bézier curves such as recurrence relations, degree elevation algorithms, the partition of unity property, linear independence (polynomial basis), recursive evaluation algorithms, and a (q,h)-Marsden identity. This work extends and generalizes some analogous results of Goldman, Simeonov, and Zafiris [2,4,9,10] for q- and h-Bernstein polynomials and q- and h-Bézier curves. Most of our proofs will use the method of mathematical induction (with respect to the polynomial degree), instead of the blossoming techniques used by Goldman, Simeonov, and Zafiris [2,4,9,10], since we lack the machinery of the (q,h)-blossoming theory. The advantage of our approach is that we can establish all these important properties almost from scratch using only the very popular and well-understood induction argument, instead of the much less familiar theory of quantum blossoming.

We begin with some notation and terminology. Let $g(t)=qt+h$ be a linear function, $q\neq 0,-1$. The j-th composition of the function g is defined by

$$g^{[j]}(t)=(\underbrace{g\circ g\circ\cdots\circ g}_{j \text{ times}})(t),\quad j\geq 1.$$

We set $g^{[0]}(t)=t$. For example

$$\begin{aligned} &g^{[2]}(t)=(g\circ g)(t)=g(g(t))=q^2t+qh+h,\\ &g^{[3]}(t)=g(g^{[2]}(t))=q\cdot g^{[2]}(t)+h=q^3t+(q^2+q+1)h. \end{aligned}$$

Notice that

$$q+1=\frac{1-q^2}{1-q}=[2]_q \quad \text{and} \quad q^2+q+1=\frac{1-q^3}{1-q}=[3]_q,$$

where $[n]_q=\frac{1-q^n}{1-q}$ if $q\neq 1$, $[n]_q=n$ if $q=1$ and $[n]_0=1$ are the so-called q-integers [6]. By induction it is easy to show that

$$g^{[n]}(t)=q^n\cdot t+\left(\frac{1-q^n}{1-q}\right)h=q^nt+[n]_qh. \tag{1.1}$$

The (q,h)-Bernstein polynomials of degree n on the interval $[a,b]$ are defined by [3]

$$B_k^n(t;[a,b];q,h)=\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{\prod_{j=0}^{k-1}(t-g^{[j]}(a))\cdot\prod_{j=0}^{n-k-1}(b-g^{[j]}(t))}{\prod_{j=0}^{n-1}(b-g^{[j]}(a))}, \tag{1.2}$$