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A Note on the Operator Equation Generalizing the Notion of Slant Hankel Operators

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Abstract. The operator equation $\lambda M_{\bar{z}} X = X M_{z^k}$, for $k \geq 2$, $\lambda \in \mathbb{C}$, is completely solved. Further, some algebraic and spectral properties of the solutions of the equation are discussed.

Key Words: Hankel operators, slant Hankel operators, generalized slant Toeplitz operators, generalized slant Toeplitz operators, spectrum of an operator.

AMS Subject Classifications: 47B35

1 Introduction

Let $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{T}$ denote the set of integers, set of complex numbers and the unit circle, respectively. Let $L^2(\mathbb{T})$ (simply written as L^2) denote the classical Hilbert space with standard orthonormal basis $\{e_n : n \in \mathbb{Z}\}$, where $e_n(z) = z^n$ for each $z \in \mathbb{T}$. The symbol H^2 denotes the space generated by $\{e_n : n \geq 0\}$. The symbol L^∞ is used to denote the space of all essentially bounded measurable functions on $\mathbb{T}$ and $H^\infty = L^\infty \cap H^2$. The theory of Hankel operators, which is a beautiful area of mathematical analysis, admits of vast applications. In 1861, Hankel [12] began the study of finite matrices whose entries depend only on the sum of the coordinates and such objects are called Hankel matrices. In 1881, Kronecker [14] obtained first theorem about infinite Hankel matrices that characterizes Hankel matrices of finite rank.

The development of the theory of Hankel operators led to different generalizations of the original concept, like, slant Hankel operators, λ-Hankel operators and (λ, μ)-Hankel operators (see [2, 5] and [8]). A lot of progress has taken place in the study of Hankel operators on Bergman spaces on the disk, Dirichlet type spaces, Bergman and Hardy spaces on the unit ball in C^n or on symmetric domains, etc. [16].

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Hankel operators on the space L^2 are characterized by the operator equation $M_{\overline{z}}X = XM_z$, whereas on the Hardy space H^2 these are characterized by the operator equation $U^*X = XU$, where U is the forward unilateral shift operator on the Hardy space H^2. We refer to [10, 11, 16] and the references therein for the basic study of Hankel operators on these spaces. Motivated by the approach initiated by Barría and Halmos [2], various equations, like, $M_{\overline{z}}X = XM_{z^2}$ (solutions of which are named as slant Hankel operators [2]), $U^*X - XU = \lambda X$, $\lambda \in \mathbb{C}$ (solutions of which are named as λ-Hankel operators [5]) etc. are attained by mathematicians. In this row, generalized slant Hankel operators [3] have also been obtained which are nothing but the solution of the operator equation $M_{\overline{z}}X = XM_{z^k}$, for $k \geq 2$ and are named as k^{th}-order slant Hankel operators. Work of Avendaño [5] dragged our attention to the operator equation $\lambda M_{\overline{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. Clearly, for $\lambda = 1$, this equation characterizes the k^{th}-order slant Hankel operators and if further $k = 2$ then it is nothing but the equation characterizing slant Hankel operators.

From the work of Nehari [15], it is known that each Hankel operator is induced by an essentially bounded measurable symbol $\phi \in L^\infty$ and is denoted as H_ϕ. Not much is known about spectral properties of Hankel operators in terms of the inducing symbol. Power [17] described the essential spectrum of H_ϕ for piecewise continuous functions $\phi \in L^\infty$. In this paper, we completely solve the operator equation $\lambda M_{\overline{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. We describe some of the spectral properties of the solutions of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. We achieve the containment of a closed disc in the spectrum of each non-zero operator satisfying the equation $\lambda M_{\overline{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$.

2 Operator equation: $\lambda M_{\overline{z}}X = XM_{z^k}$ for $k \geq 2$, $\lambda \in \mathbb{C}$

In last two decades various operator equations generalizing the notion of Hankel operators have been discussed, for the details and importance of which we suggest the references [2, 3] and [4]. The purpose here is to call attention to the operator equation $\lambda M_{\overline{z}}X = XM_{z^k}$, for an integer $k \geq 2$ and $\lambda \in \mathbb{C}$. Throughout the paper, k is assumed to be an integer greater than or equal to 2. We begin with the following result.

Theorem 2.1. *The only solution of the operator equation* $\lambda M_{\overline{z}}X = XM_{z^k}$, $|\lambda| \neq 1$ *is the zero operator.*

Proof. Suppose that X satisfies $\lambda M_{\overline{z}}X = XM_{z^k}$. First, consider the case $|\lambda| < 1$. Define a map $\tau : \mathfrak{B}(L^2) \rightarrow \mathfrak{B}(L^2)$ as $\tau(X) = \lambda M_{\overline{z}} X M_{\overline{z}^k}$. Then $\|\tau\| \leq |\lambda| < 1$ and $(I - \tau)$ is invertible. Now $(I - \tau)X = 0$, which implies that $X = 0$.

Now consider the case $|\lambda| > 1$. This time we define the mapping τ as $\tau(X) = M_z X M_{z^k}$. Now $\|\tau\| \leq 1$ so $(\lambda I - \tau)$ is invertible and this provides that $X = 0$. This completes the proof. $\square$