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# Oscillatory Strongly Singular Integral Associated to the Convex Surfaces of Revolution

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**Abstract.** Here we consider the following strongly singular integral

$$T_{\Omega,\gamma,\alpha,\beta}f(x,t)=\int_{R^n} e^{i|y|^{-\beta}}\frac{\Omega(\frac{y}{|y|})}{|y|^{n+\alpha}}f(x-y,t-\gamma(|y|))dy,$$

where $\Omega\in L^p(S^{n-1})$, $p>1$, $n>1$, $\alpha>0$ and $\gamma$ is convex on $(0,\infty)$.

We prove that there exists $A(p,n)>0$ such that if $\beta>A(p,n)(1+\alpha)$, then $T_{\Omega,\gamma,\alpha,\beta}$ is bounded from $L^2(R^{n+1})$ to itself and the constant is independent of $\gamma$. Furthermore, when $\Omega\in C^\infty(S^{n-1})$, we will show that $T_{\Omega,\gamma,\alpha,\beta}$ is bounded from $L^2(R^{n+1})$ to itself only if $\beta>2\alpha$ and the constant is independent of $\gamma$.

**Key Words**: Oscillatory strongly rough singular integral, rough kernel, surfaces of revolution.

**AMS Subject Classifications**: 42B20, 42B35

## 1 Introduction

The standard Hilbert transform along a curve is defined as

$$H_\Gamma f(x)=P.V.\int_{-1}^{1} f(x-\Gamma(t))\frac{dt}{t},$$

where $\Gamma:(-1,1)\to R^n$ is a continuous curve in $R^n$. The study of these operators was initiated by Fabes and Rivière [7]. In [18], Stein and Wainger proved that $H_\Gamma$ is bounded on $L^p(1<p<\infty)$ if $\Gamma$ is well-curved in $R^n$. Here we say that $\Gamma$ is well-curved, if $\Gamma$ is smooth with $\Gamma(0)=0$ and a segment of the curve containing the origin lies in a subspace of $R^n$ spanned by

$$\frac{d^{(k)}\Gamma(t)}{dt}\Big|_{t=0},\quad k=1,2,\cdots.$$

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When $n=2$, $\Gamma(t)$ can be written as $(t,\gamma(t))$. If $\gamma(t)$ is flat at the origin, i.e.,

$$\frac{d^{(k)}\gamma(t)}{dt}\Big|_{t=0}=0$$

for $k=1,2,\cdots$, then it is easy to see that $\Gamma(t)$ is not well-curved in $R^2$. The main contributions on the Hilbert transforms along the flat curves were made by Wainger and his colleagues. Readers can see [1,12–15,19,21,22] among numerous references, in particular, the good survey papers [18] and [23].

Another interesting operator in harmonic analysis is the hyper Hilbert transform

$$H_\alpha f(x)=P.V.\int_{-1}^{1} f(x-t)\frac{dt}{t|t|^\alpha},\quad 0<\alpha<1.$$

We know that the operator $H_\alpha$ is bounded from the Sobolev space $L^p_\alpha$ to the Lebesgue space $L^p$ for $1<p<\infty$, because of the mean zero of the kernel of $H_\alpha$. One naturally expected that, without the assumption of the mean zero on the kernel, the worsened singularity of $H_\alpha$ near the origin can be counterbalanced by an oscillatory factor $e^{i|t|^{-\beta}}$ $(\beta>0)$ as $t$ approaches zero. This idea motivated the study of the oscillatory hyper Hilbert transforms (see [10]) and the strongly singular integral operators in high dimensional spaces. More details of the strongly singular integral operators can be found in [8, 9, 16, 20].

Consider the following oscillatory hyper Hilbert transforms,

$$H_{\Gamma,\alpha,\beta}f(x)=\int_{-1}^{1} f(x-\Gamma(t))e^{i|t|^{-\beta}}\frac{dt}{t|t|^\alpha},\quad \alpha,\beta\geq 0,$$

where $\Gamma(0)=0$, $\beta>\alpha$.

Zielinski [24] studied the $L^2-$boundedness of $H_{\Gamma,\alpha,\beta}$ along the parabola $(t,t^2)$. In [2], for $\Gamma(t)=(t,|t|^q), q\geq 2$, Chandarana proved that $H_{\Gamma,\alpha,\beta}$ is bounded on $L^2$ if and only if $\beta\geq 3\alpha$. When $n=3$, a similar result was proved in [3].

In [4] and [5], we generalized these results in $R^n$ and removed all assumptions on the indexes. At the same time, Laghi and Lyall in [11] proved that if $\Gamma(t)$ is well-curved, then $H_{\Gamma,\alpha,\beta}$ is bounded on $L^2(R^n)$ if and only if $\beta\geq(n+1)\alpha$.

In [6] we study the general case $\Gamma(t)=(t,\gamma(t))$ in $R^2$ where $\gamma$ is flat on $(0,1)$. We obtain some interesting results. In the same paper, we construct some examples to illustrate the complexity of this problem.

Here we consider the following oscillatory strongly singular integral associated to the surfaces of revolution,

$$T_{\Omega,\gamma,\alpha,\beta}f(x,t)=\int_{R^n} e^{i|y|^{-\beta}}\frac{\Omega(\frac{y}{|y|})}{|y|^{n+\alpha}}f(x-y,t-\gamma(|y|))dy. \tag{1.1}$$